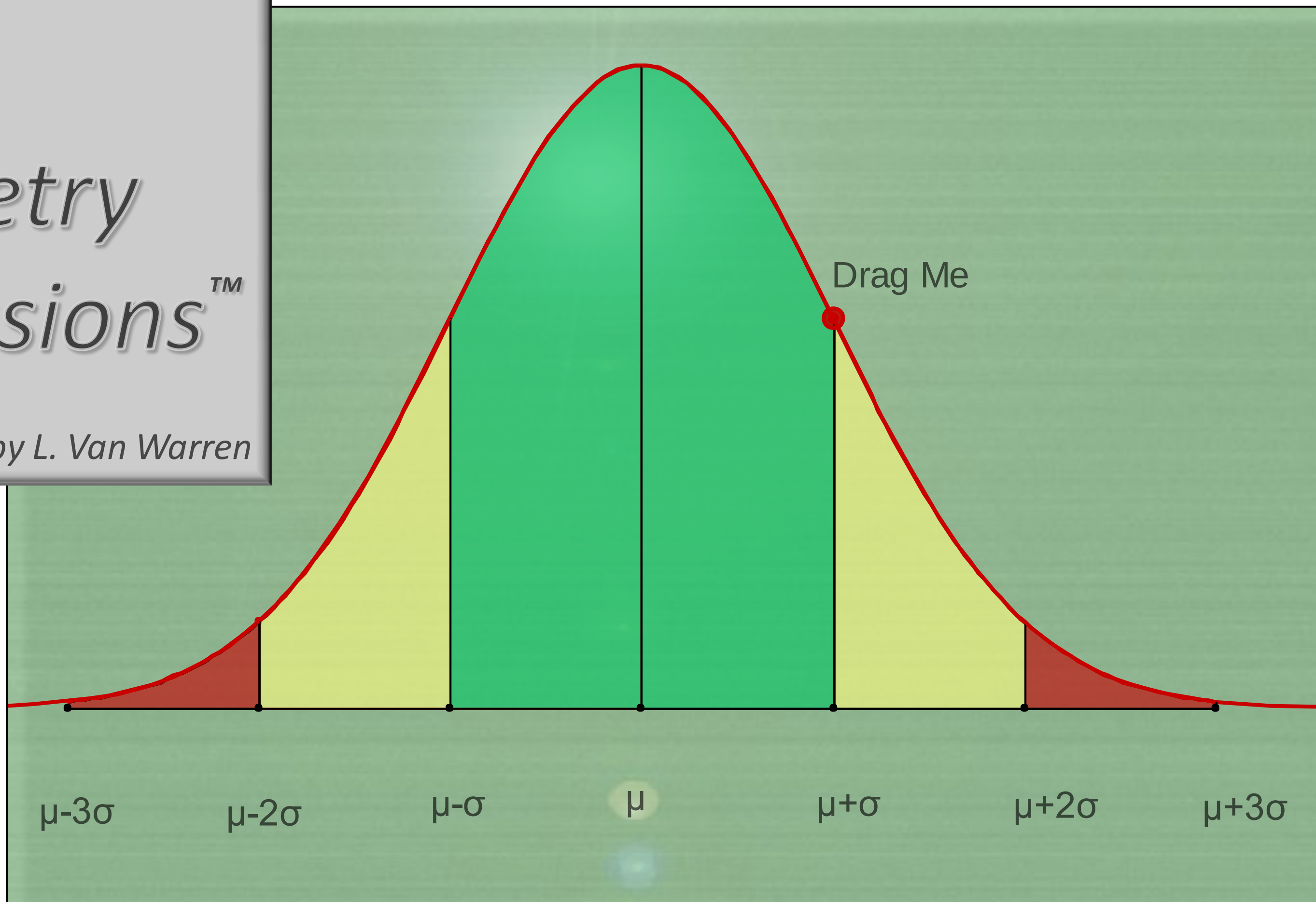
Learning Calculus With Geometry Expressions™

by L. Van Warren

Lecture 21: The Definite Integral



Chapter 5: Integration

LECTURE	TOPIC
19	*ANTIDERIVATIVES*
20	*INTEGRATION: AREA AND DISTANCE*
21	***THE DEFINITE INTEGRAL***
22	*FUNDAMENTAL THEOREM OF CALCULUS*

Calculus Inspiration

Courtesy: Institute for Advanced Study

Edward Witten

PhD. Princeton

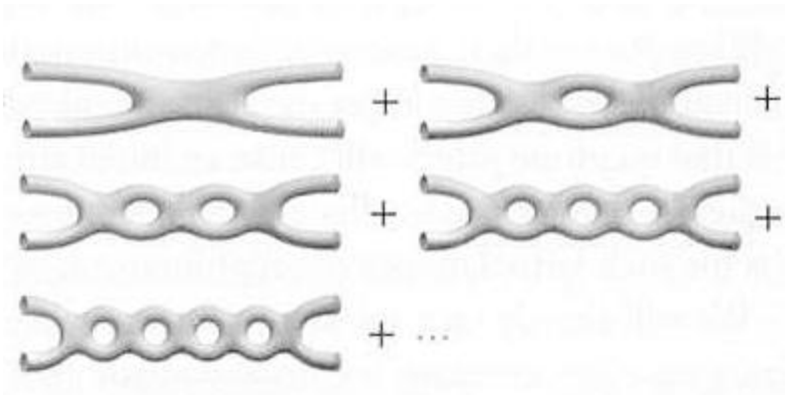
- M Theory
- String Theory
- Quantum Theory

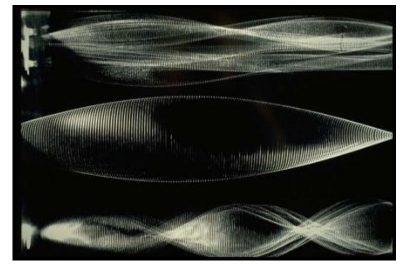
- Fields Medal (1990)
- Poincare Prize (2006)
- Crafoord Prize (2008)

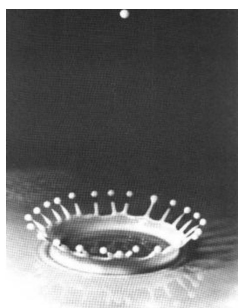
- A Living Einstein

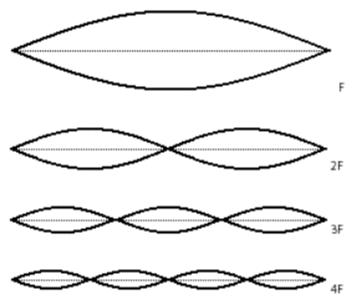
Genealogy is David Gross → Geoffrey Chew → Enrico Fermi

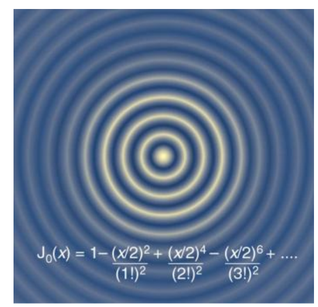
Strings and Things



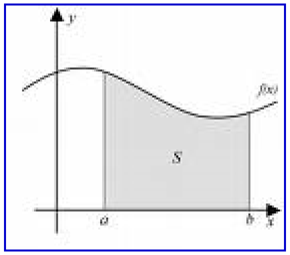








Hot and Cold Running Integrals:



... of an **definite integral** is the ...
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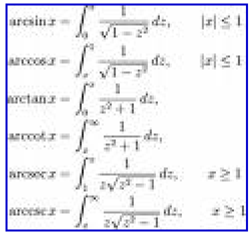
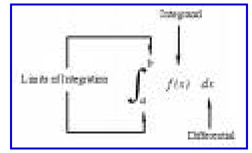


... function as a **definite integral**:



... is the **definite integral**:

... of **definite integral** continued
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definite integral

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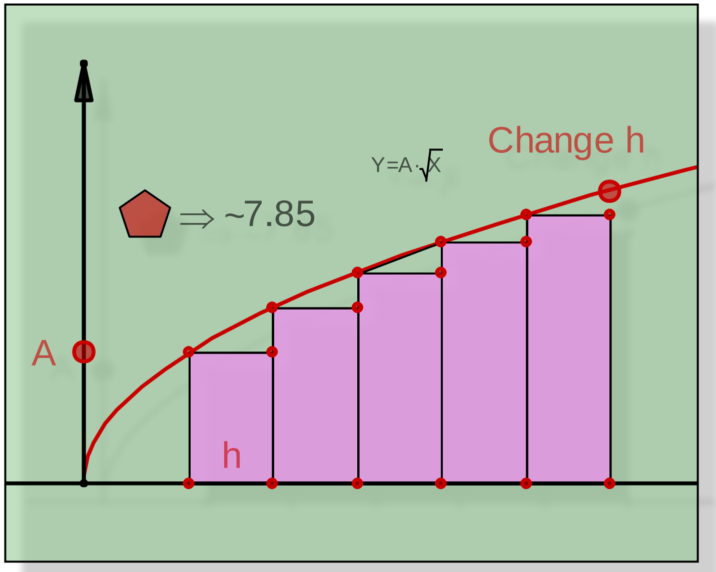
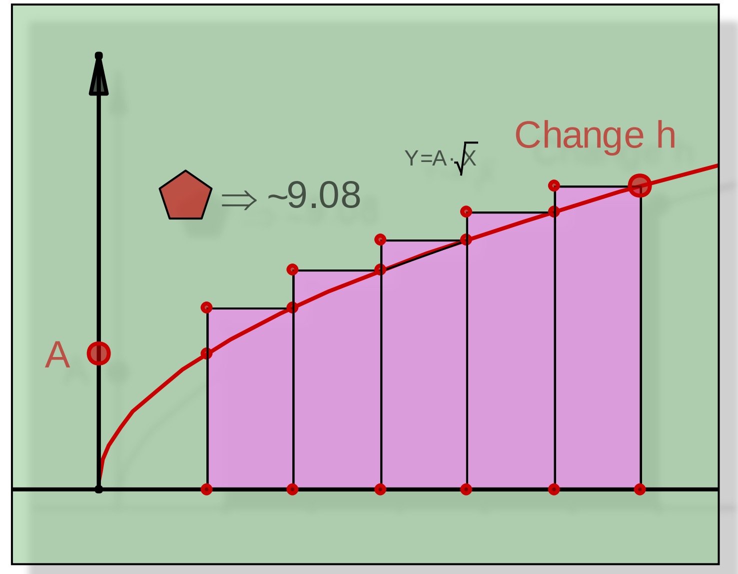
Definite Integral Review:

Lecture21-ADefiniteIntegral.gx
Lecture21-BDefiniteIntegral.gx





Left Riemann Sum:

$$A_L = f(x_0)\cdot h + f(x_1)\cdot h + \ldots + f(x_{n-1})\cdot h$$
$$= f(x_0) \quad + f(x_1) \quad + \ldots + f(x_{n-1}) \cdot h$$
$$= \sum_{i=0}^{n-1} f(x_i)\cdot h$$

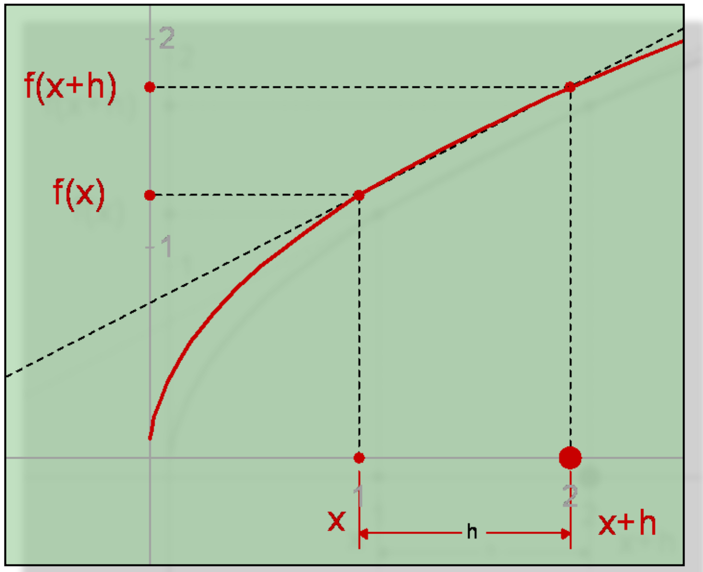
Right Riemann Sum:

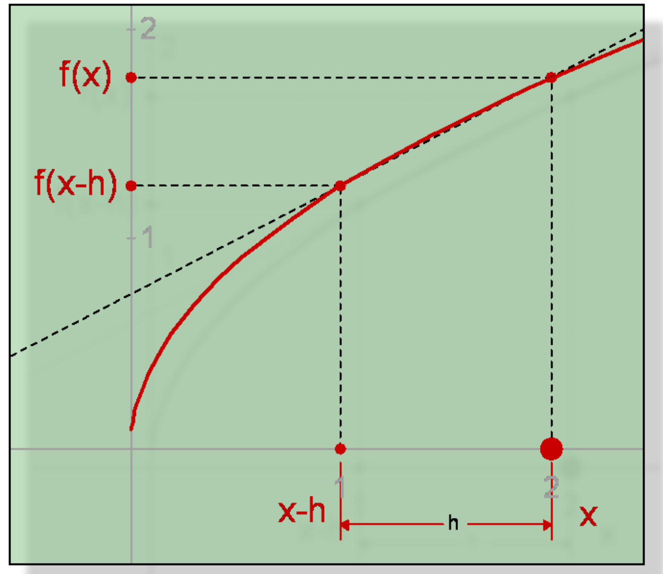
$$A_R = f(x_1)\cdot h + f(x_2)\cdot h + \ldots + f(x_n)\cdot h$$
$$= f(x_1) \quad + f(x_2) \quad + \ldots + f(x_n) \cdot h$$
$$= \sum_{i=1}^{n} f(x_i)\cdot h$$

Derivative By Limit:

Lecture21-ADerivativeByLimit.gx
Lecture21-BDerivativeByLimit.gx

The derivative can be defined:





From the left:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

LAGRANGE NOTATION

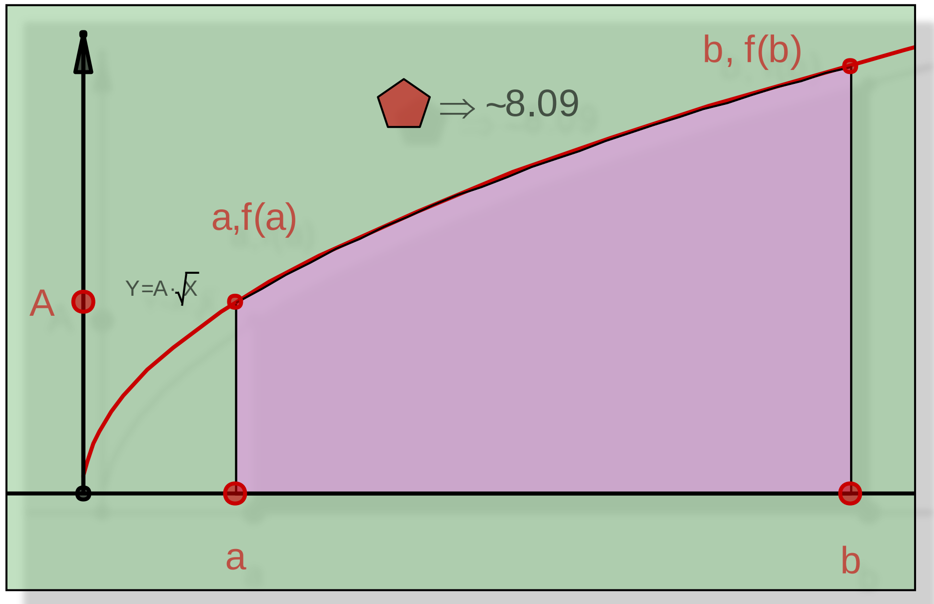
Or from the right:

$$f'(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

We have the same option with the definition of the integral.

Definite Integral By Limit:

Lecture21-IntegralByLimit.gx



Left Limit of Sum:

$$F(x)=\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i^*)\cdot h$$

$$=\lim_{n\to\infty}\left[f(x_0^*)+f(x_1^*)+\ldots+f(x_{n-1}^*)\right]\cdot h$$

$$where \quad x_i^* \;\; is \;\; anywhere \;\; in \; [x_{i-1}..x_i]$$

Right Limit of Sum:

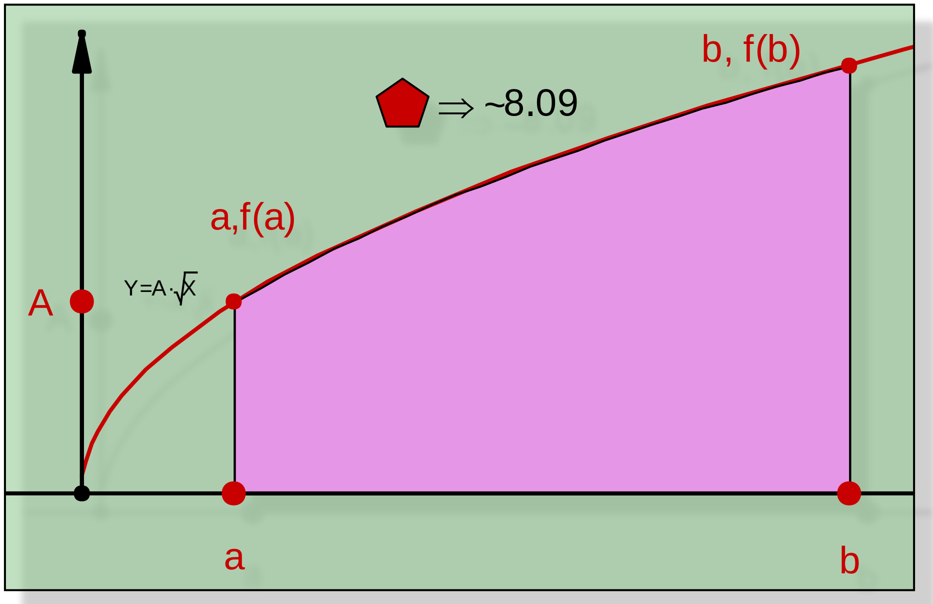
$$F(x)=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\cdot h$$

$$=\lim_{n\to\infty}\left[f(x_1^*)+f(x_2^*)+\ldots+f(x_n^*)\right]\cdot h$$

$$where \quad x_i^* \;\; is \;\; anywhere \;\; in \; [x_i..x_{i+1}]$$

Definite Integral Definition:

Lecture21-IntegralByLimit.gx



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\cdot h$$

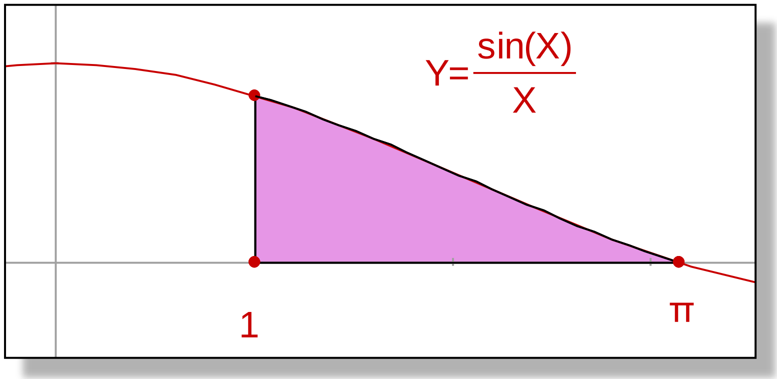
$$where \quad h = \Delta x = \frac{b-a}{n}$$

If this limit exists, then f(x) is integrable on x = [a..b]

Limitations of Definite Integral:

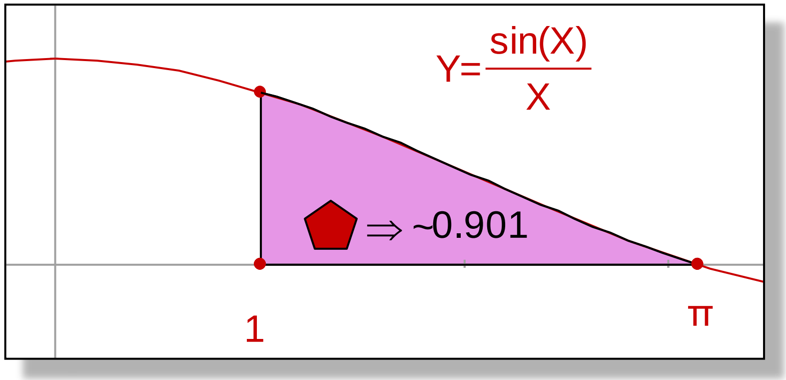
Lecture21-SincFunction.gx

Point 1: Many functions for which the limit exists have not yet been integrated in closed form:



$$\int_1^{\pi} \frac{\sin x}{x} = ?$$

Point 2: Many functions for which the limit exists have stable numerical solutions:



Precise Meaning of Definite Integral Limit:

For every number ε > 0, there is an integer N such that:

$$\left| \int_a^b f\ x\ dx - \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\cdot \Delta x \right| < \varepsilon$$

For every integer n > N and for every choice of x_i^* in $[x_{i-1}...x_i]$.

Integrability Theorem:

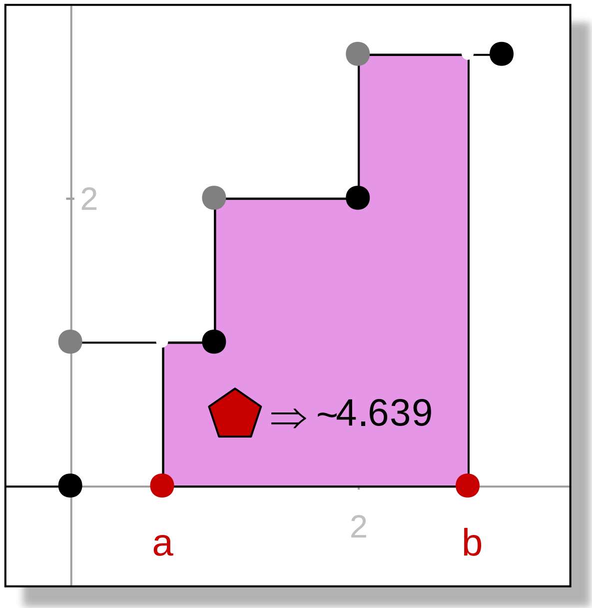
Lecture21-IntegralOfCeiling.gx

IF

f(x) is continuous on x in [a..b] OR
f(x) has only a finite number of jump discontinuities

THEN

$$\int_a^b f\ x\ dx \quad \textit{exists!}$$



First Choice for x_i^*:

If we choose $x_i^* = x_i$ then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Provided the limit exists.

In this case: $$\Delta x = \frac{b-a}{n}$$

and $$x_i = a + i\Delta x$$

Translating Limit to Definite Integral:

Lecture21-TranslatingLimit.gx

For example express:

$$\lim_{n\to\infty}\sum_{i=1}^{n} x_i^3 + x_i \sin\ x_i \quad \Delta x$$

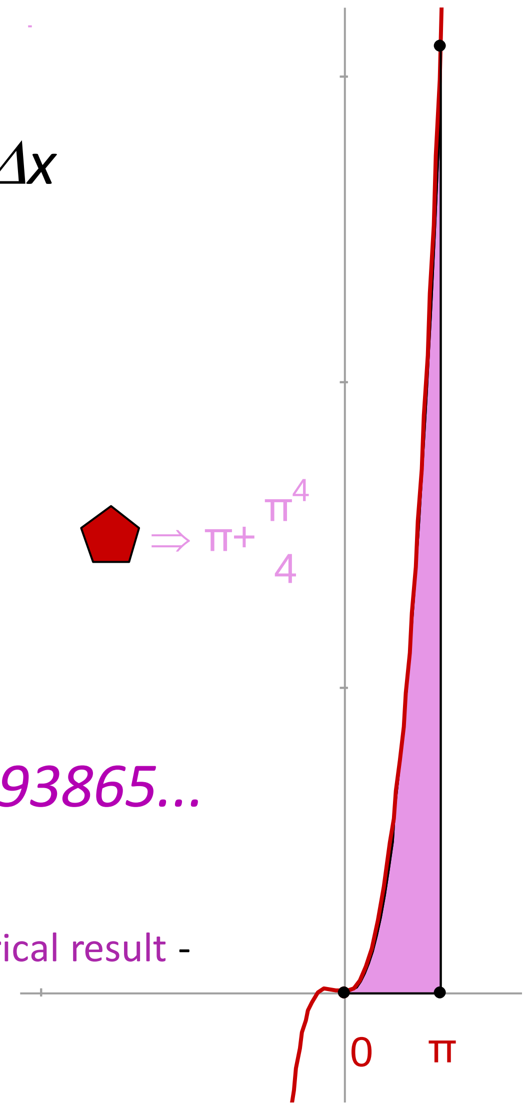
As an integral on the interval [0..π].

By the defintion we have:

$$\int_0^{\pi} x^3 + x\sin\ x \quad dx =$$

$$\frac{\pi^4}{4} + \pi \qquad \cong \quad 27.493865...$$

$\Rightarrow \pi + \frac{\pi^4}{4}$



Exercise: Show that the definite integral evaluates to the numerical result -

1) By hand.
2) Using wxMaxima™.
3) Using Geometry Expressions™.

Example: Finite Riemann Sum:

For example consider:

$$\sum_{i=1}^{3} x_i^3 - 6x \quad \Delta x$$

We seek the area on the interval [0..3] with $\Delta x = 1$ *and* $x_i = i$.
Writing the terms of the summation we have:

$$x_1^3 - 6x_1 \quad \Delta x + \quad x_2^3 - 6x_2 \quad \Delta x + \quad x_3^3 - 6x_3 \quad \Delta x \quad =$$

$$1^3 - 6 \cdot 1 \quad \cdot 1 \quad + \quad 2^3 - 6 \cdot 2 \quad \cdot 1 \quad + \quad 3^3 - 6 \cdot 3 \quad \cdot 1 \quad =$$

$$1 - 6 \qquad + \quad 8 \; - 12 \qquad + \quad 27 - 18 \qquad =$$

$$-5 \qquad + \qquad -4 \qquad + \qquad 9 \qquad = \; 0$$

Exercise:

1) Draw the function using Geometry Expressions™ and discuss why the Riemann sum is zero.
2) Compute the definite integral in wxMaxima™. Is zero the correct answer for the area?

Some Sums

Lecture21-SomeSums.wxm

Summation	Result
$\sum_{i=1}^{n} i$	$\frac{n^2+n}{2}$
$\sum_{i=1}^{n} i^2$	$\frac{2n^3+3n^2+n}{6}$
$\sum_{i=1}^{n} i^3$	$\frac{n^4+2n^3+n^2}{4}$

Exercise: Using wxMaxima™ compute the next entry in the table.

Note 1 on Notation:

The definite integral:

$$\int_a^b f\,\,x\,\,dx$$

does not depend on x! You get the same answer in each of:

$$\int_a^b f\,\,x\,\,dx = \int_a^b f\,\,t\,\,dt = \int_a^b f\,\,s\,\,ds$$

The result is not a number, but rather a function of the limit variables.

Note 1 on Notation - Example:

Consider the case of a limit variable:

$$\int_0^x y\,dy = \left.\frac{y^2}{2}\right|_0^x = \frac{x^2}{2} - \frac{0^2}{2} = \frac{x^2}{2}$$

Versus a limit constant:

$$\int_0^1 y\,dy = \left.\frac{y^2}{2}\right|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$

The first expression is a function, the second is a number.

If you think about it, functions can be numbers and vica versa.

Note 2 on Notation:

Some claim that: $\int_a^b f \;\; x \;\; dx$ is "all one symbol".

Consider functional notation:

$$integrate \;\; f \;\; x \;\; , x, a, b \;\; ;$$

which has the same meaning!

The latter is a "functional" with four arguments.
It is a "functional" because one of the arguments is itself a function.

But any function can be a functional that accepts functions as an argument because we routinely compose functions as in f(g(x)).

So limits, derivatives and integrals are all just functions.

Note 3 on Notation – Improper Integral:

If the limits a or b of the definite integral $\int_a^b f\ x\ dx$ are $+\infty$ or $-\infty$,

we say the integral is "improper".

This is something of a misnomer since the limit can still exist.
For example, if b is $+\infty$ we write:

$$\int_a^{\infty} f\ x\ dx = \lim_{b\to\infty} \int_a^b f\ x\ dx$$

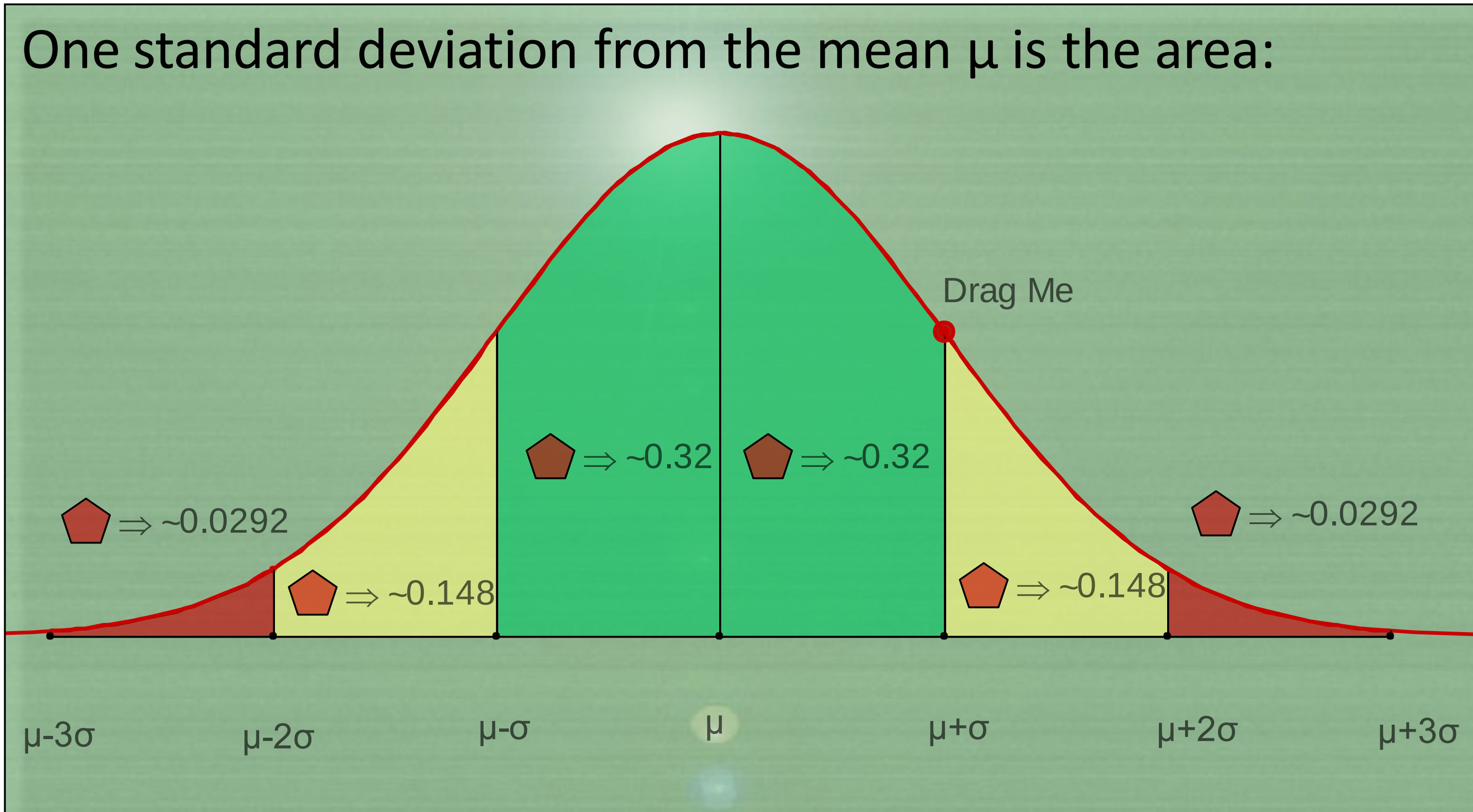
Consider the case of the unnormalized sinc function:

$$\int_a^{\infty} \frac{\sin\ x}{x} dx = \lim_{b\to\infty} \int_0^b \frac{\sin\ x}{x} dx = \frac{\pi}{2}$$

Paradoxically, *sinc(x)* cannot be integrated for finite limits!

Definite Integral Application - Statistics:

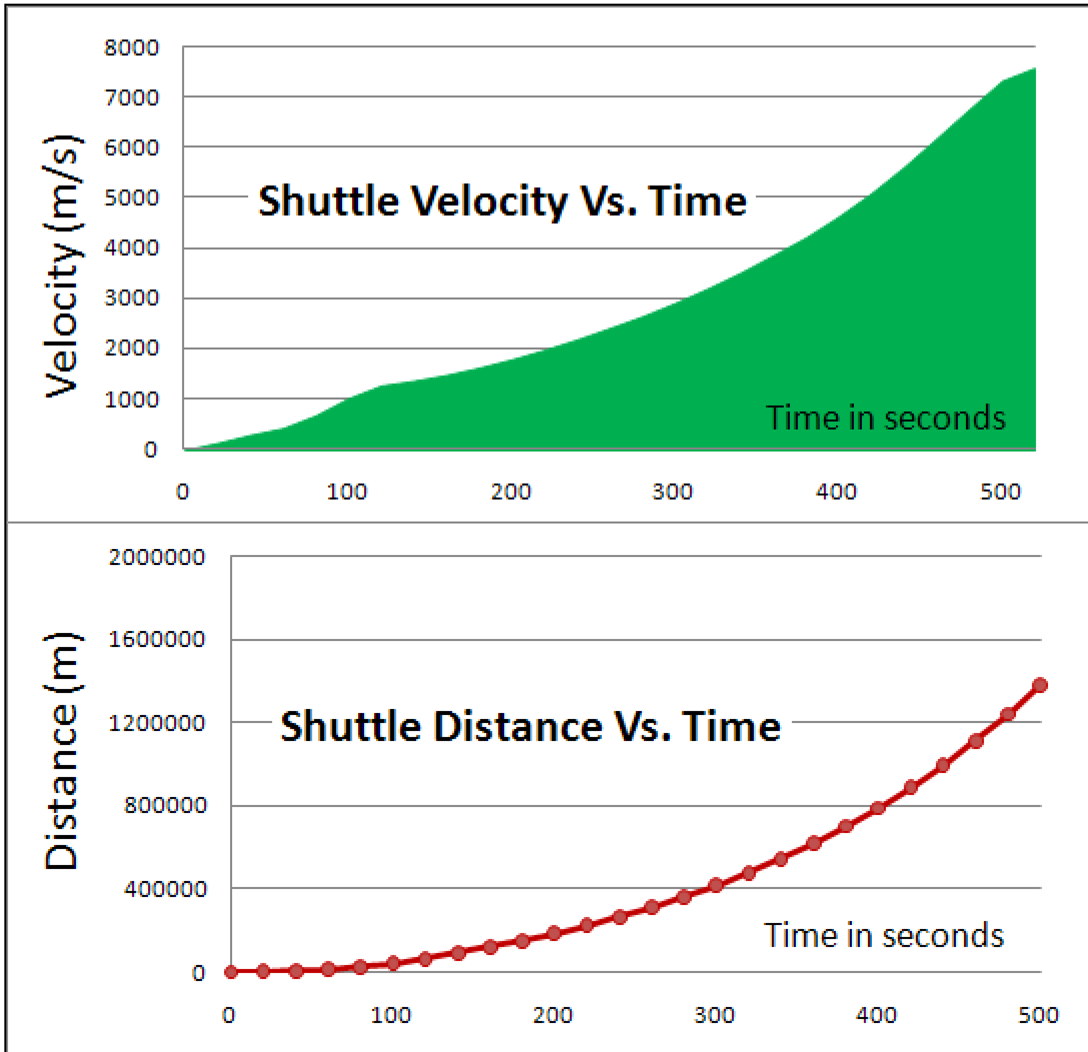
Lecture21-StandardDeviation.gx



Computed by the definite integral: $\int_{\mu-\sigma}^{\mu+\sigma} \frac{e^{-\frac{x-\mu}{2\sigma^2}^2}}{\sqrt{2\pi\sigma^2}}\,dx = 0.682$

Definite Integral Application - Flight:

Lecture21-IntegrateVelocity.xls



Integrate velocity over time to obtain distance travelled.

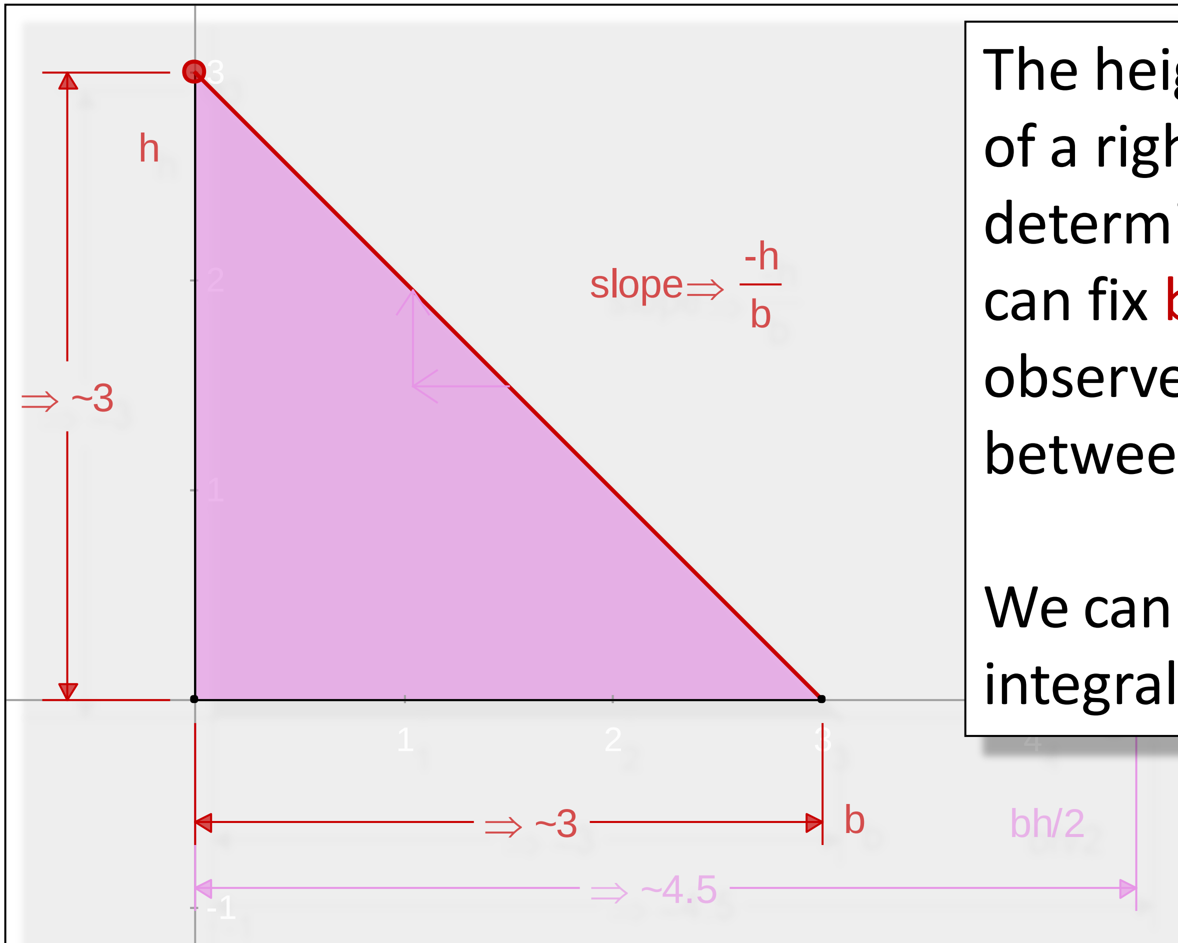
Computed numerically using trapezoidal rule.

Exercises:

1) Open spreadsheet.
2) Use left Riemann sum instead of trapezoidal rule.

Application – Area of a Right Triangle:

Lecture21-TriangleArea.gx



The height h and base b of a right triangle determine its area. We can fix b and vary h to observe relationships between size and area.

We can use the definite integral to find the area.

Exercise: Drag h and bh/2 to examine size and area relationships.

Definite Integral – Area of a Right Triangle:

The triangle's area can be computed from:

$$A = \int_0^b \left[-\frac{h}{b}x + h \right] dx = \left[-\frac{h}{2b}x^2 + hx \right] \Bigg|_{x=0}^{x=b} = \frac{bh}{2}$$

The general form for the area under any curve is:

$$A = \int_a^b f\ x\ \ dx$$

Exercise:

1) Find the area of a rectangle using the integral approach.
2) Find the area of the function y=400 sin(400x) between x = 1 and x = π.
3) What is the meaning of negative area?

Definite Integral – Area of a Right Triangle:

Lecture21-TriangleArea.wxm

Even for simple integrals, symbolic integration reduces errors and speeds discovery:

```
f(x):=(-h/b)*x+h;
```

$$f(x):=\frac{-h}{b}x+h$$

```
integrate(f(x),x);
```

$$h\,x-\frac{h\,x^2}{2\,b}$$

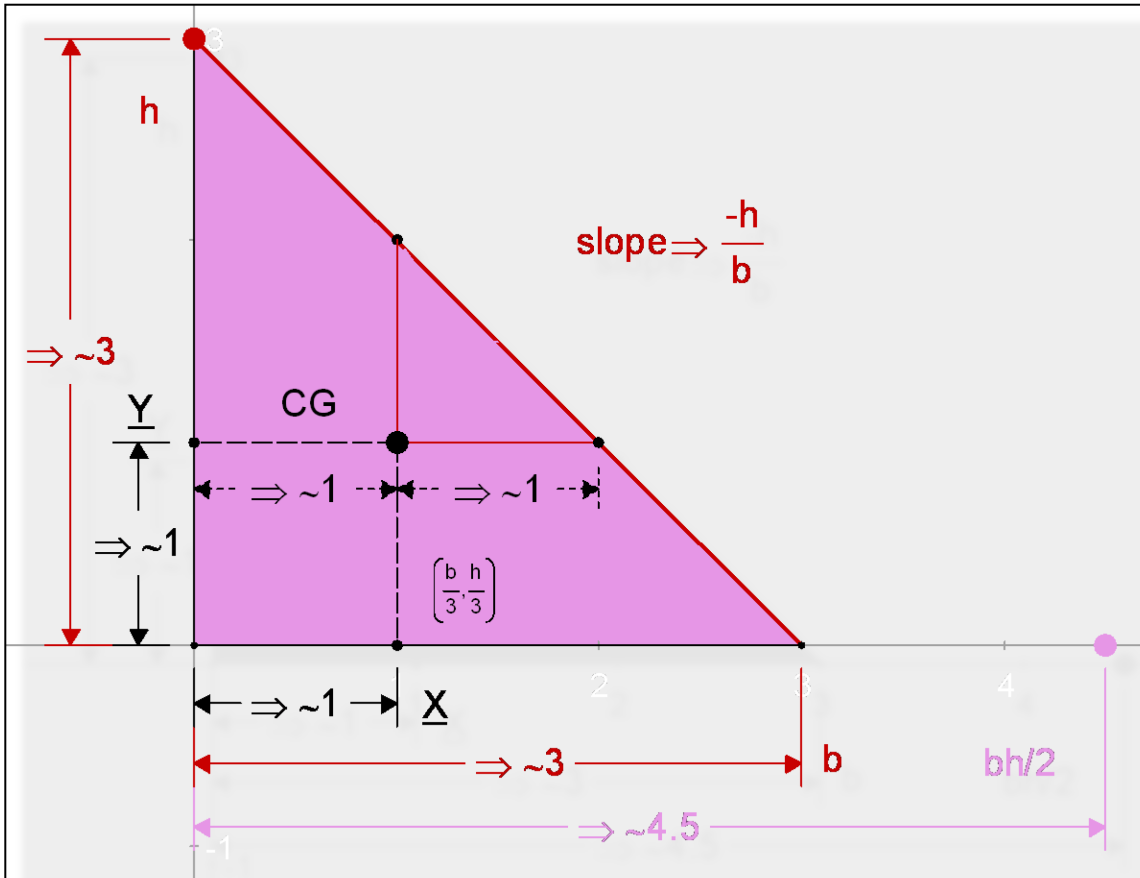
```
integrate(f(x),x,0,b);
```

$$\frac{b\,h}{2}$$

1) f(x) is the roof line of the triangle.
2) Perform indefinite integration. We could evaluate this result by hand between 0 and b, OR
3) Use the definite integral to obtain the classic result for the area of a triangle.

Application - Center of Gravity:

Lecture21-TriangleCG.gx



The CG ($\underline{X}$, $\underline{Y}$) of a triangle is ⅓ the distance from each base.

But the CG locally splits the triangle in half!

Explain.

Exercise: Drag h and examine the Area and CG relationships.

Definite Integral – Center of Gravity:

The x-coordinate of the triangle's CG, X, is:

$$\underline{X} = \frac{\int_0^b x\left[-\frac{h}{b}x + h\right]dx}{\int_0^b \left[-\frac{h}{b}x + h\right]dx} = \left[-\frac{h}{2b}x^2 + hx\right]\Bigg|_{x=0}^{x=\underline{b}} = \frac{b}{3}$$

X is the x-axis station at which the triangle would balance on a knife edge. The general form is:

$$\underline{X} = \frac{\int_a^b x \cdot f\ x\ \ dx}{\int_a^b f\ x\ \ dx}$$

Exercise:

1) Show by hand that the next to last step produces the result shown.
2) Generate an expression for and compute Y.

Definite Integral – Center of Gravity:

Lecture21-TriangleCG.wxm

For complex integrals, symbolic integration is much faster.

```
f(x):=(-h/b)*x+h;
```

$$f(x):=\frac{-h}{b}x+h$$

1) f(x) is the roof line of the triangle.

2) Perform the indefinite integral to see the result. We could evaluate this result between 0 and b OR

```
integrate(x*f(x),x)/integrate(f(x),x);
```

$$-\frac{2\,h\,x^3-3\,b\,h\,x^2}{6\,b\left(h\,x-\frac{h\,x^2}{2\,b}\right)}$$

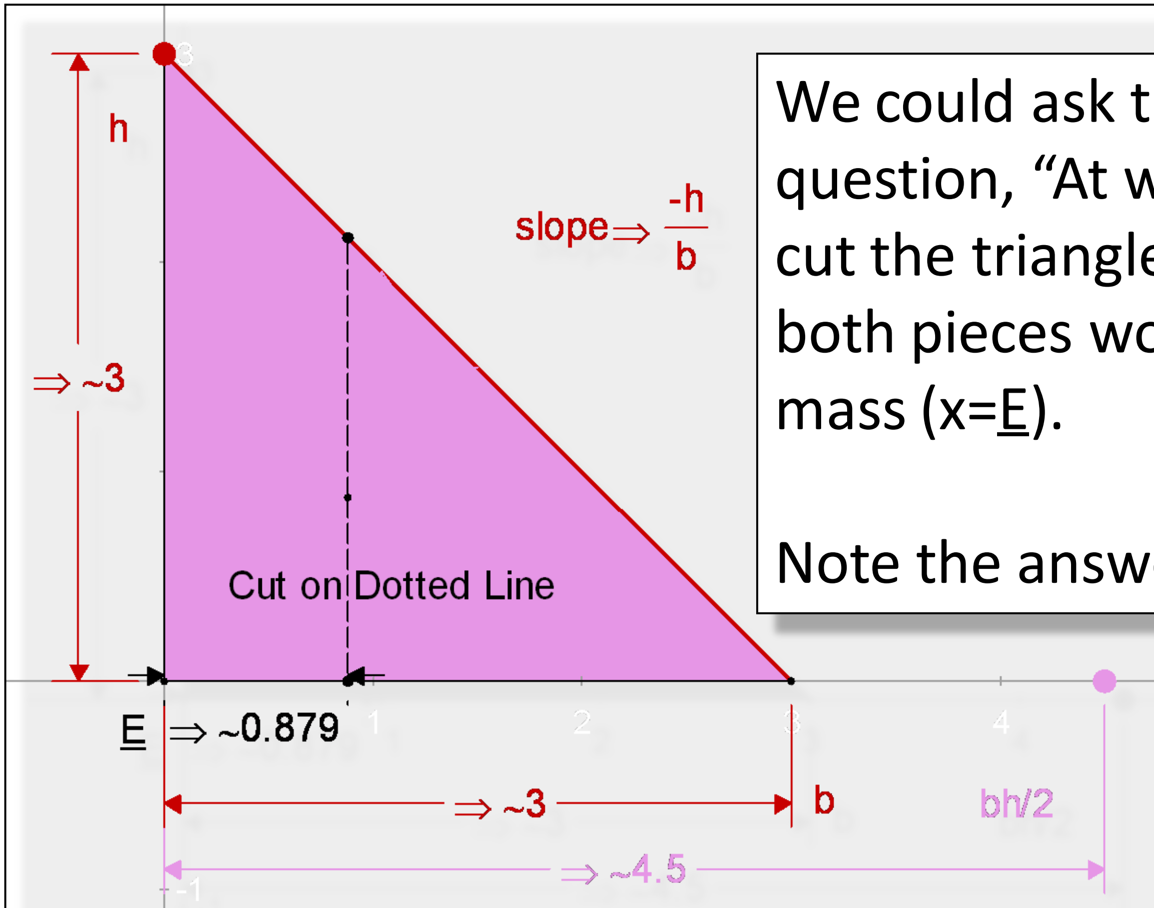
3) Repeat the expression using the definite integral. The answer is the same.

```
integrate(x*f(x),x,0,b)/integrate(f(x),x,0,b);
```

$$\frac{b}{3}$$

Application – Equal Masses:

Lecture21-TriangleEqualMasses.gx



We could ask the similar question, “At what x could we cut the triangle in two such that both pieces would have equal mass (x=E).

Note the answer is NOT the CG!

Exercise: Create an expression for a horizontal cut that would yield equal masses.

Definite Integral – Equal Masses:

The variable we want to solve for is in upper limit of our integral.

$$\int_0^{\underline{E}} \left[-\frac{h}{b}x + h \right] dx = \int_{\underline{E}}^{b} \left[-\frac{h}{b}x + h \right] dx$$

$$\left[-\frac{h}{2b}x^2 + hx \right]\Bigg|_{x=0}^{x=\underline{E}} = \left[-\frac{h}{2b}x^2 + hx \right]\Bigg|_{x=\underline{E}}^{x=b}$$

$$\underline{E} = \left[-\frac{\left(\sqrt{2}-2\right) b}{2} \right] \cong 0.29 \cdot b$$

Exercise: Find an algebraic expression for the golden ratio φ. Is there a relationship to E?

Definite Integral – Equal Masses:

Lecture21-TriangleEM.wxm

```
f(x):=(-h/b)*x+h;
```

$$f(x):=\frac{-h}{b}x+h$$

```
integrate(f(x),x,0,E)=integrate(f(x),x,E,b);
```

$$-\frac{hE^2-2bhE}{2b}=\frac{hE^2-2bhE}{2b}+\frac{bh}{2}$$

```
solve(%,E);
```

$$[E=-\frac{(\sqrt{2}-2)b}{2},E=\frac{(\sqrt{2}+2)b}{2}]$$

```
float(%);
```

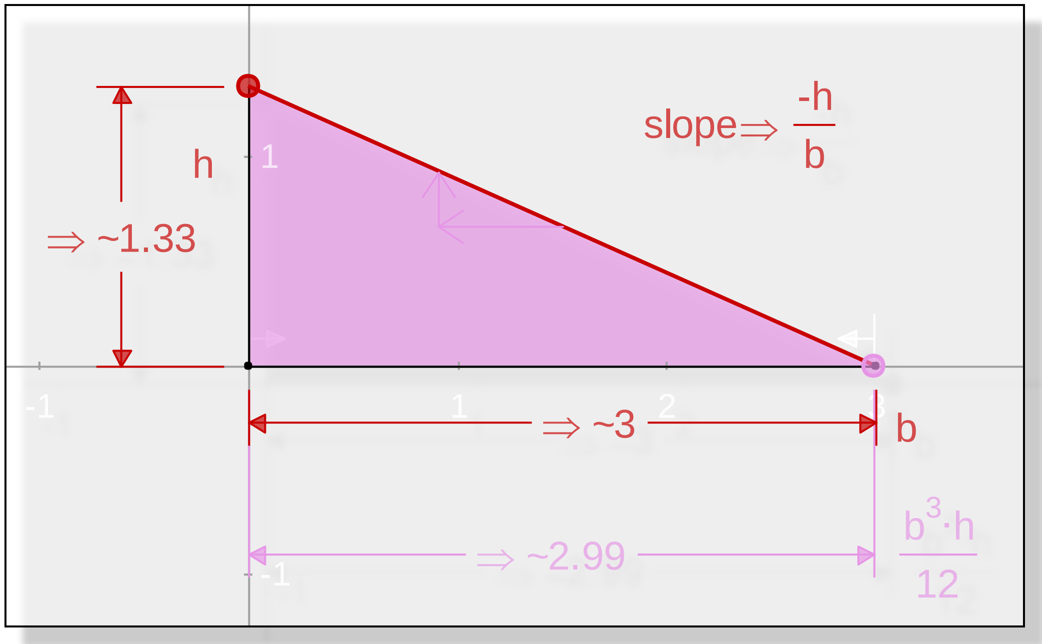
$$[E=0.29289321881345b,E=1.707106781186548b]$$

1) f(x) is the roof line of the triangle.
2) The area to the left of x=<u>E</u> equals the area to the right of x=<u>E</u>.
3) Solve the resulting expression for <u>E</u>.
4) We get two solutions, which one is plausible?
5) How much do <u>E</u> and <u>X</u> differ?

Application – Moment of Inertia:

Lecture21-TriangleMoment.gx

If a triangle was hinged on the y-axis it would oppose being swung by the distribution of its mass relative to the y-axis.



This resistance to rotation is the "moment of inertia".

Definite Integral – Moment of Inertia:

The triangle's moment of inertia can be computed from:

$$\underline{M} = \int_0^b x^2 \left[-\frac{h}{b}x + h \right] dx = \left[-\frac{hx^4}{4b} + \frac{hx^3}{3} \right] \Bigg|_{x=0}^{x=b} = \frac{b^3 h}{12}$$

The general form for the moment of inertia about the y-axis for any curve is:

$$\underline{M} = \int_a^b x^2 \cdot f\ x\ \ dx$$

Exercise:

1) Find the moment of a rectangle using the integral approach.
2) Find the moment of a semicircle with center (r,0) and radius r.
3) Can a moment of inertia be negative?

Lecture21-TriangleMoment.wxm

Definite Integral – Moment of Inertia:

```
f(x):=(-h/b)*x+h;
```

$$f(x):=\frac{-h}{b}x+h$$

```
integrate(x^2*f(x),x);
```

$$-\frac{3hx^4-4bhx^3}{12b}$$

```
integrate(x^2*f(x),x,0,b);
```

$$\frac{b^3h}{12}$$

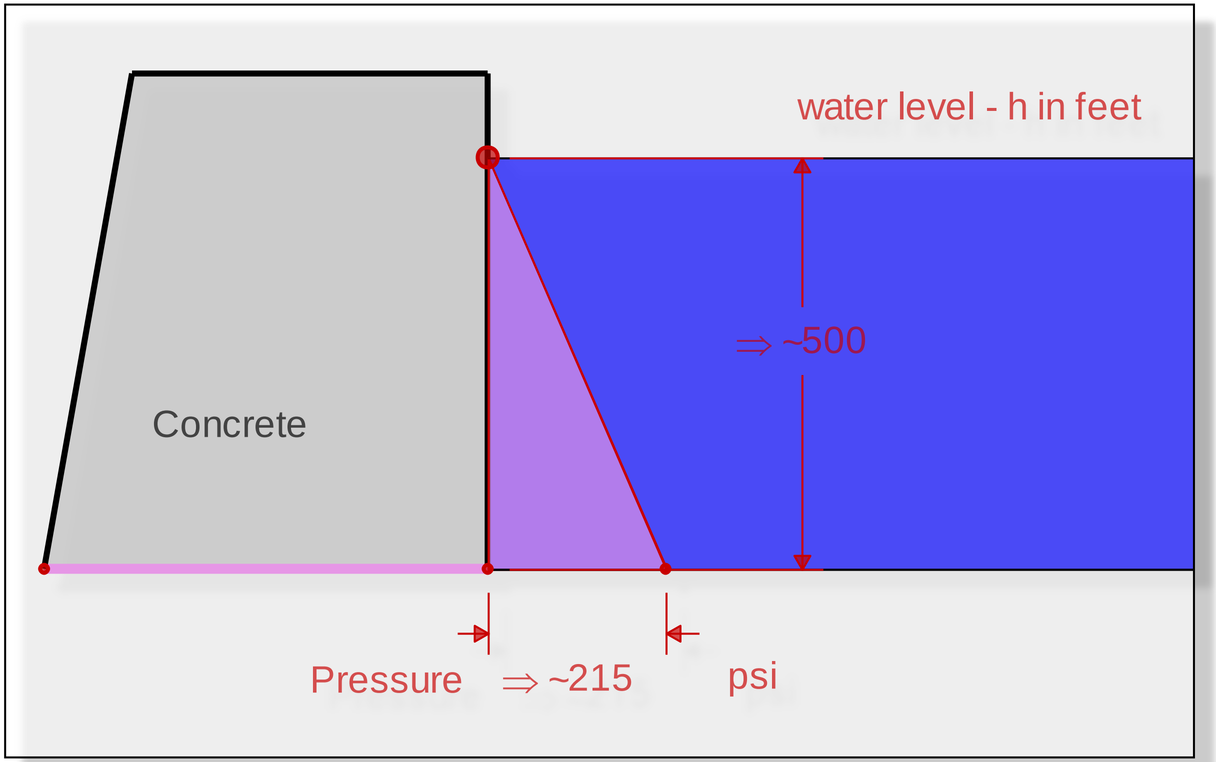
1) f(x) is the roof line of the triangle.
2) The indefinite integral is informative.
3) The moment of inertia is a simple expression for simple shapes.

Exercise: Compute the moment of inertia with respect to the x-axis.

Application – Hydrostatic Pressure:

Lecture21-WaterPressure.gx

Water pressure on dams grows linearly with depth.



Exercise: Hoover dam is 726 feet high. Use the example to estimate the pressure at the base.

Definite Integral – Hydrostatic Pressure:

The water pressure at any depth z is:

$$P(z) = \int_0^z \rho \cdot g \, dy = \rho \cdot g \int_0^z dy = \rho g y \Big|_{y=0}^{y=z} = \rho g z$$

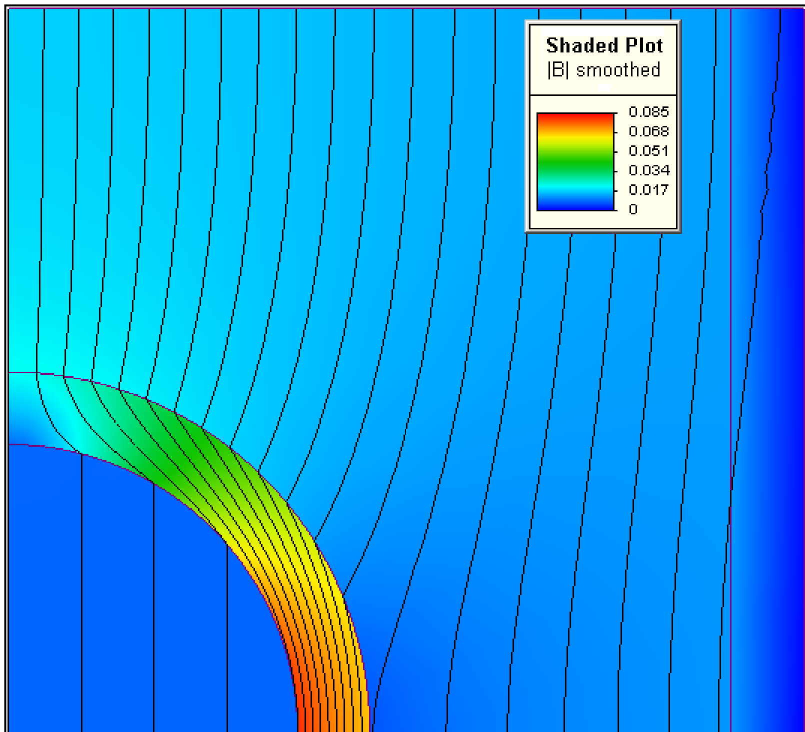
Where ρ is the density of water and g is the acceleration of gravity. Assuming these are constant we can factor them out of the integral to get a formula for pressure of depth z.

Exercise:

1) Find the centroid of the pressure triangle in the diagram and compute the value of y at which an equivalent resulting force acts.
2) Assuming that the length of the concrete base grows as the integral of the pressure, how does gray concrete area grow with water level, as h^2 or as h^3? Use symbolic calculation of the area of the concrete to justify your answer.
3) Assuming the width of the dam is constant, how does concrete cost vary as a function of the water level?

Application – Finite Element Analysis:

Definite Integrals are used to compute stress and strain in critical engineering applications.

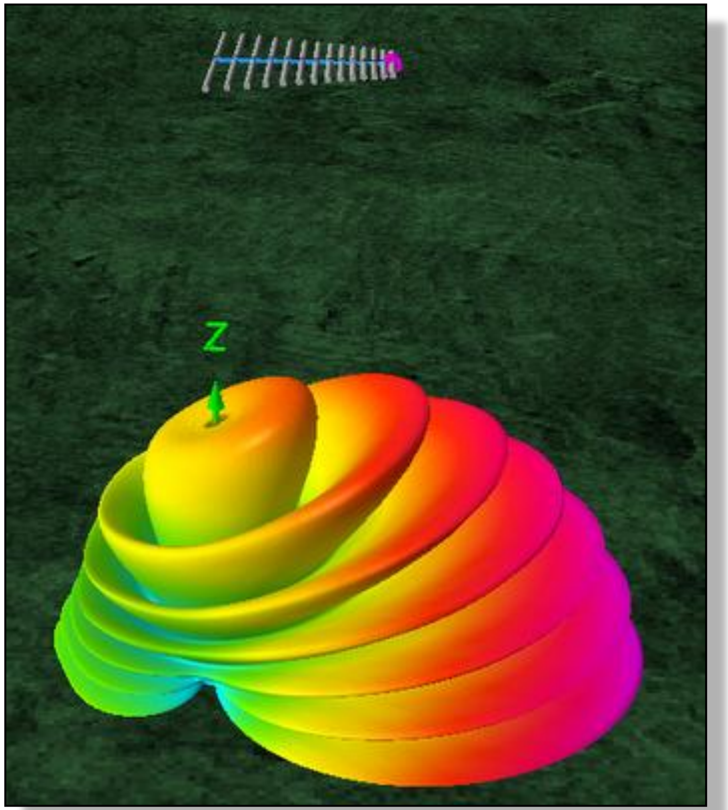


"This is a 2D solution for a cylindrically shaped magnetic shield. The exciting coil with current is shown on the extreme right as a slightly visible vertical rectangle . The lines represent the direction of calculated flux density, the colour is proportional to its magnitude ."

- Example Courtesy [Zureks]

Application – Electromagnetic Fields:

The propagation of radio frequencies are analyzed by definite integration of Maxwell's equations:



- *Computed with [4Nec2X]*

$$\oint \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

$$\oint \vec{B}\cdot d\vec{A} = 0$$

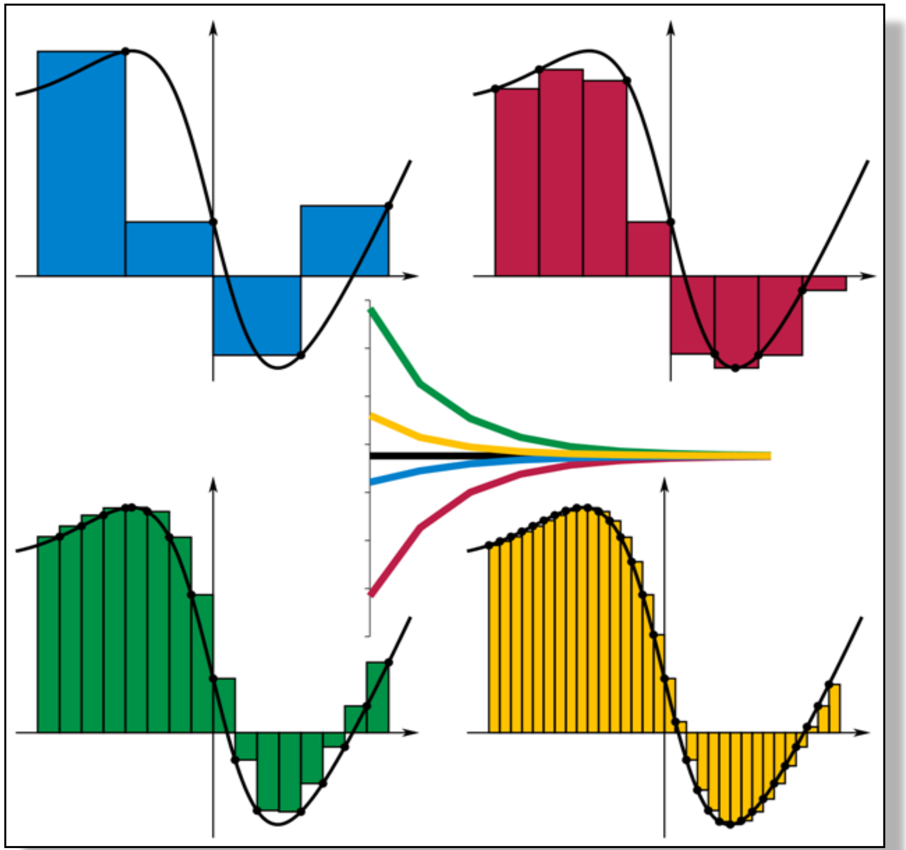
$$\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I + \varepsilon_0\mu_0\frac{d\Phi_E}{dt}$$

Exercise: Look up the meaning of each of these symbols.

This image shows the antenna gain pattern for a herringbone antenna.
The red lobes point in the direction of maximum gain for transmitting and receiving.

Rate of Convergence:



The exact value of an integral can be approximated to any desired degree of accuracy by increasing the value of n in the expression:

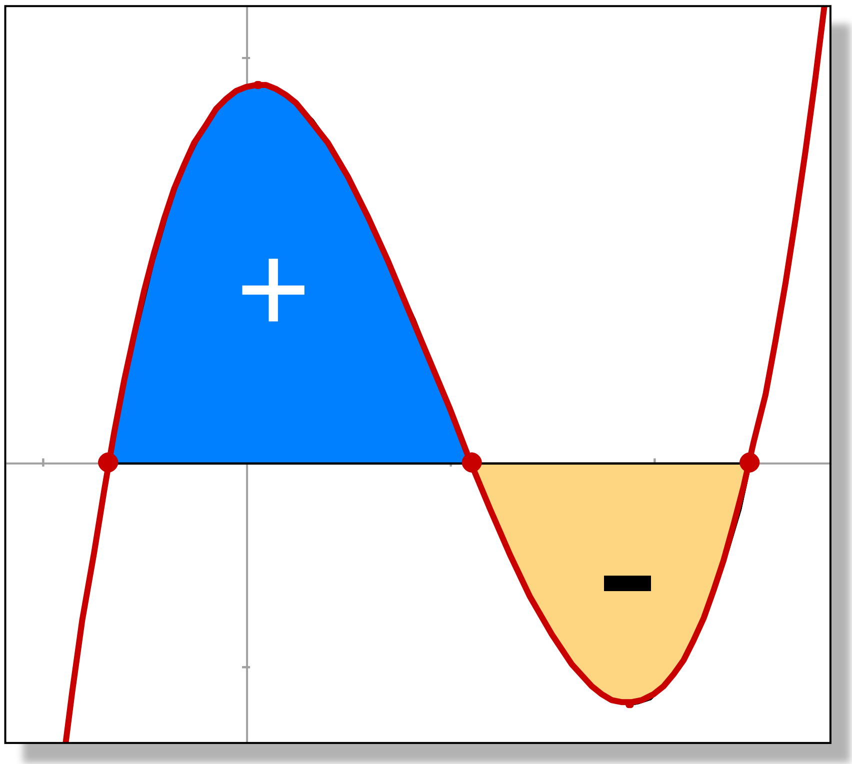
$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$$

The rate at which an approximation approaches the exact value, as n increases, is called the rate of convergence.

Different approximations converge at different rates.

Positive and Negative Area:

Lecture21-PositiveAndNegativeArea.gx



When we integrate to find the area under a curve, area that is above the x-axis will return positive results, with areas greater than zero.

When the curve goes below the x-axis, the area is negative.

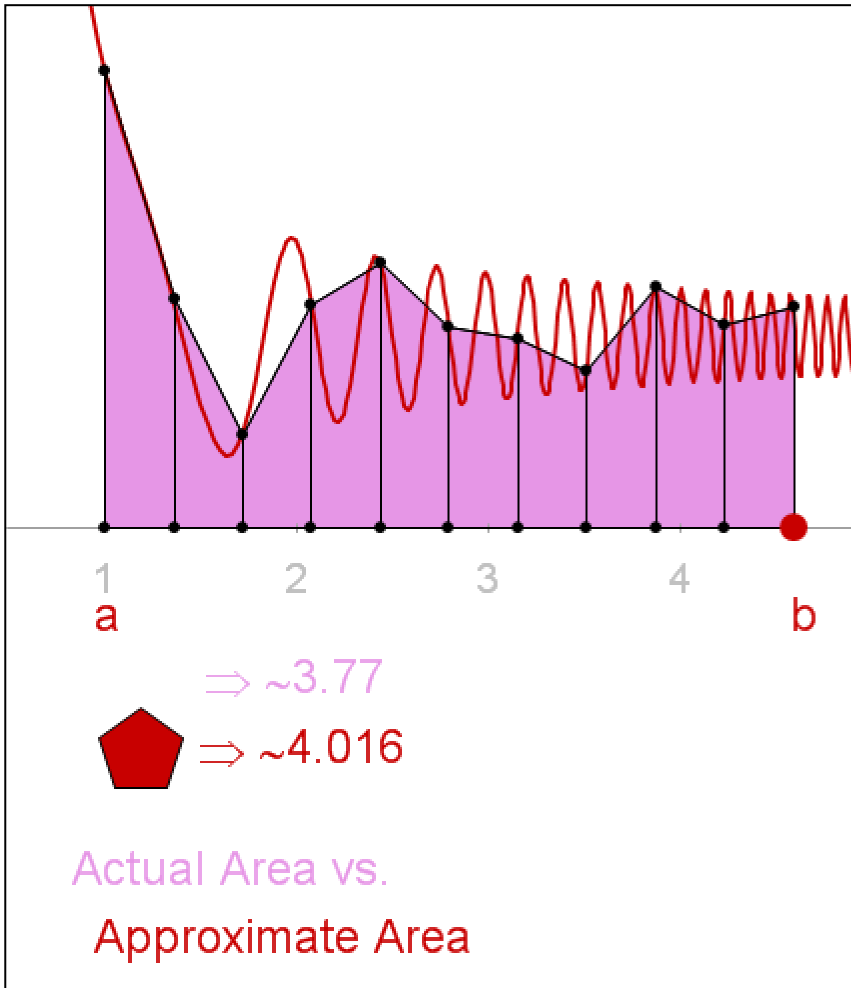
This can be written:

$$\int_a^b f(x)\,dx = A_+ + A_-$$

$$= |A_+| - |A_-|$$

Stiff Systems:

Lecture21-StiffSystems.gx



When a function changes slowly in one interval and rapidly in another we say the function is "stiff".

Stiff functions appear in vibration and impact problems.

William Gear developed integration techniques for stiff systems.

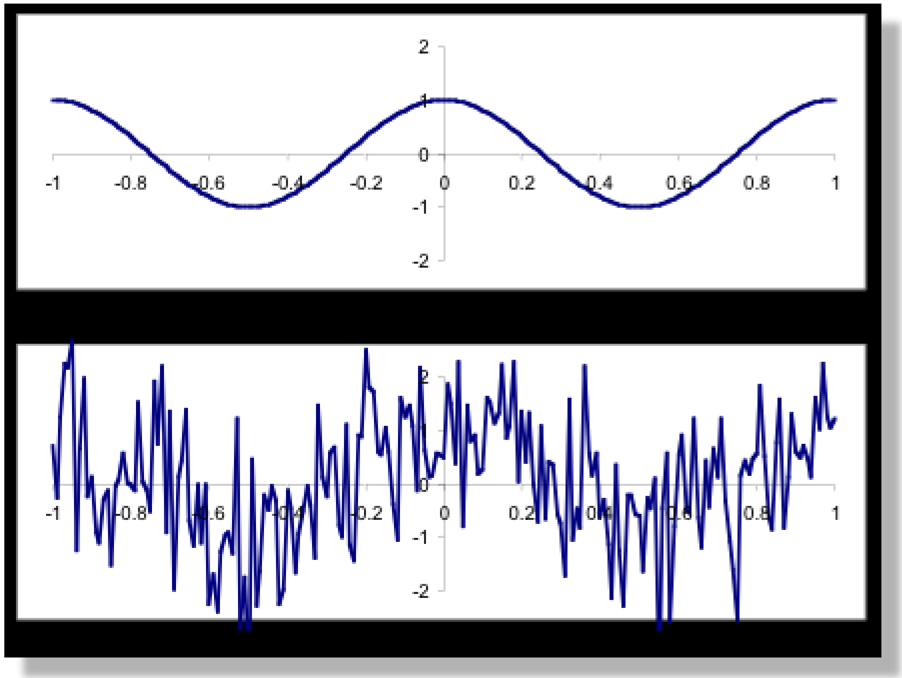
Numerical integration is called "quadrature".

Adaptive Quadrature samples the intervals correctly using recursion.

Exercise:

1) Drag b to show the weakness of uniform sampling.
2) Which region is undersampled?
3) Which is oversampled?

Noisy Systems:



In the presence of noise, an approximate solution can be more accurate than the “exact” solution and often easier to find.

With noisy data integration tends to reduce the error.

Differentiated data with noise is worthless for decision making.

```
h[i,j]:=j*x*sin(x^i);
```

$$h_{i,j} := j\,x\,\sin\left(x^{i}\right)$$

```
genmatrix(h,3,3);
```

$$\begin{bmatrix} x\sin(x) & 2\,x\sin(x) & 3\,x\sin(x) \\ x\sin\left(x^2\right) & 2\,x\sin\left(x^2\right) & 3\,x\sin\left(x^2\right) \\ x\sin\left(x^3\right) & 2\,x\sin\left(x^3\right) & 3\,x\sin\left(x^3\right) \end{bmatrix}$$

End

```
integrate(%,x);
```

$$\begin{bmatrix} \sin(x)-x\cos(x) & 2\left(\sin(x)-x\cos(x)\right) & 3\left(\sin(x)-x\cos(x)\right) \\ -\dfrac{\cos\left(x^2\right)}{2} & -\cos\left(x^2\right) & -\dfrac{3\cos\left(x^2\right)}{2} \\ \int x\sin\left(x^3\right)dx & \int 2\,x\sin\left(x^3\right)dx & \int 3\,x\sin\left(x^3\right)dx \end{bmatrix}$$