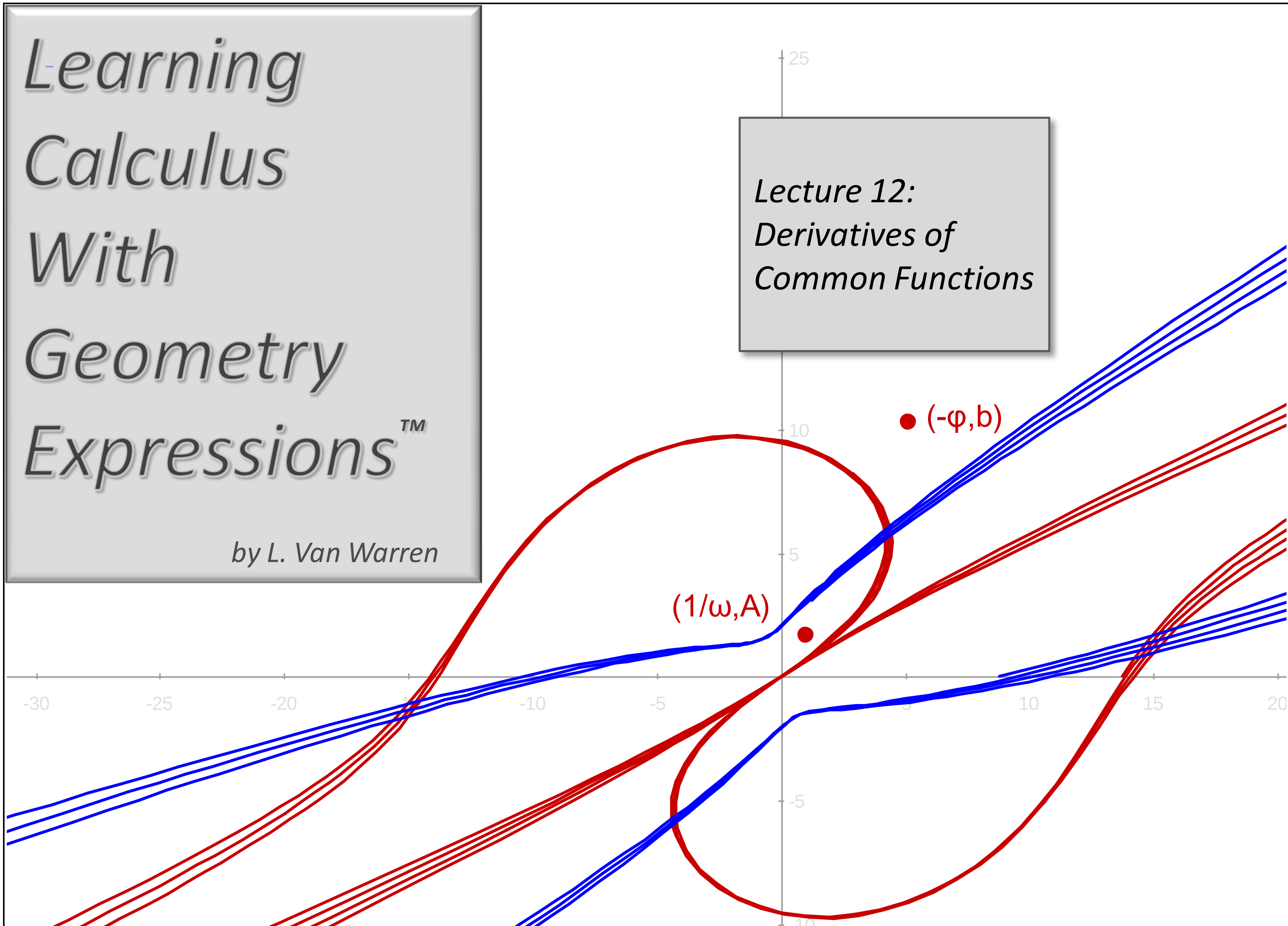
Learning Calculus With Geometry Expressions™

by L. Van Warren

Lecture 12: Derivatives of Common Functions



Chapter 3: Derivatives

Lecture	Topic
10	Definition of the Derivative
11	Properties of the Derivative
12	**Derivatives of Common Functions**
13	Implicit Differentiation

Calculus Inspiration

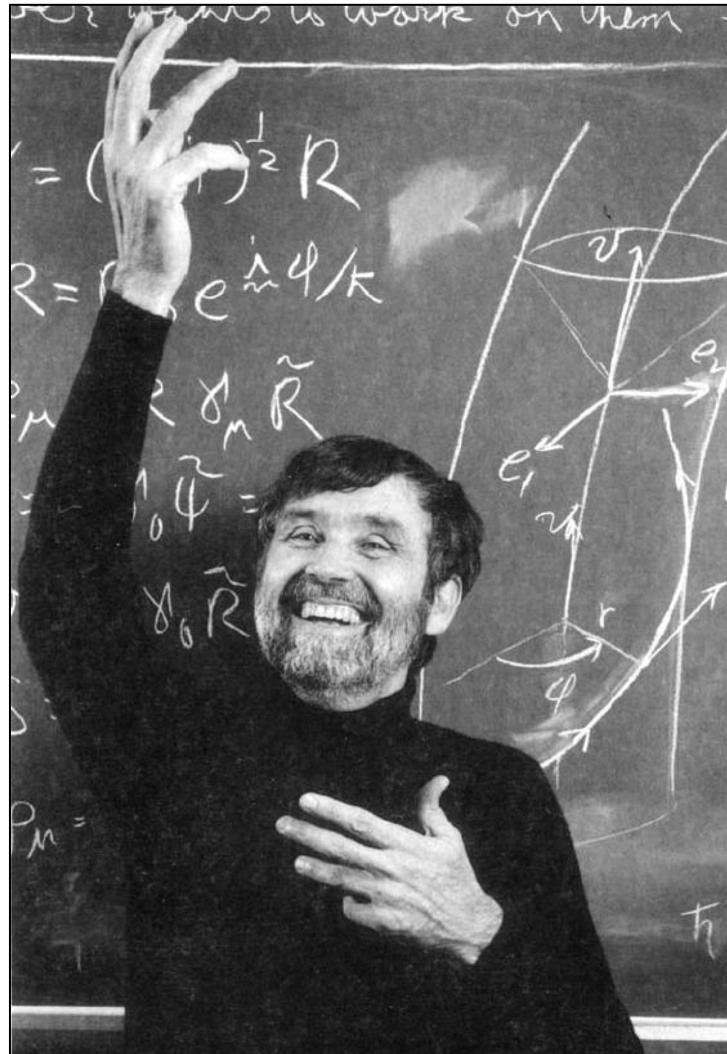
David Hestenes

PhD. UCLA

- Geometric Algebra
- Geometric Calculus
- Relativity and Electron Theory

- Oersted Medal (2002)

- Cognitive Research in Science Education



-Portrait by Jeff Wilson
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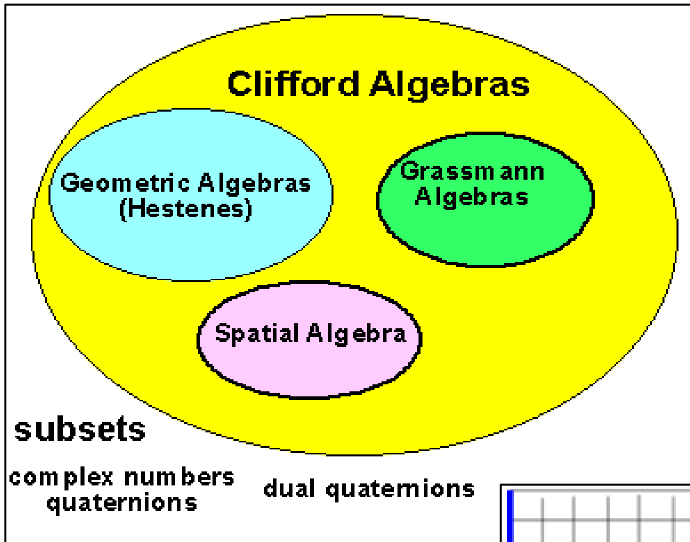
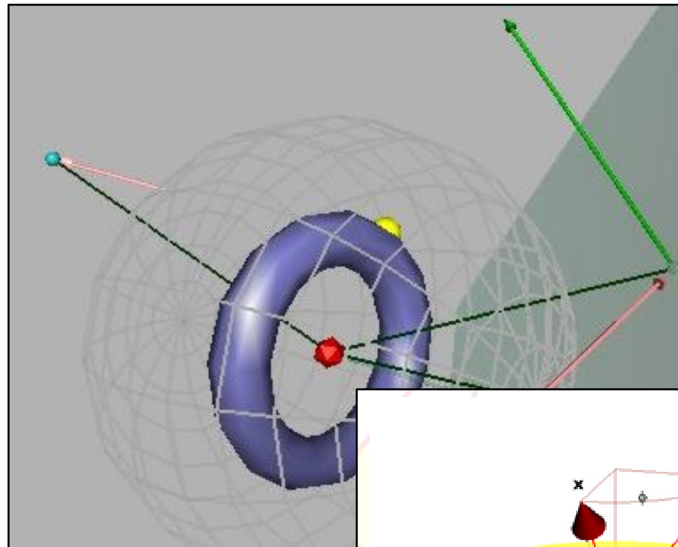
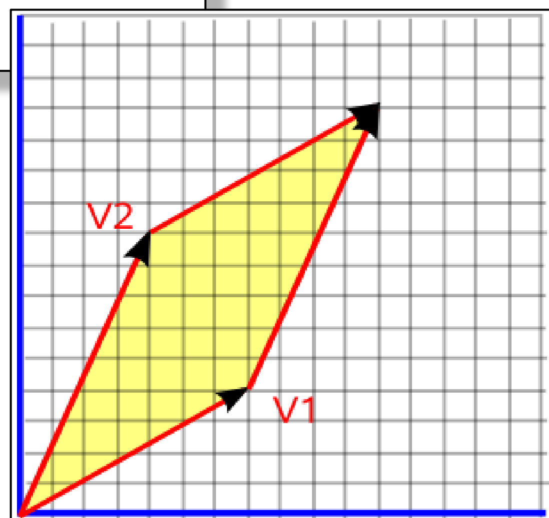
Geometric Algebras







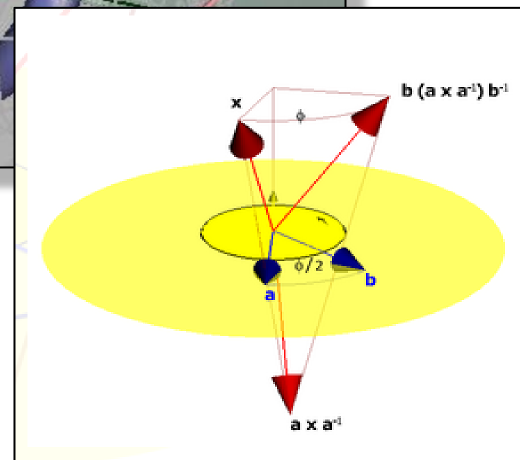
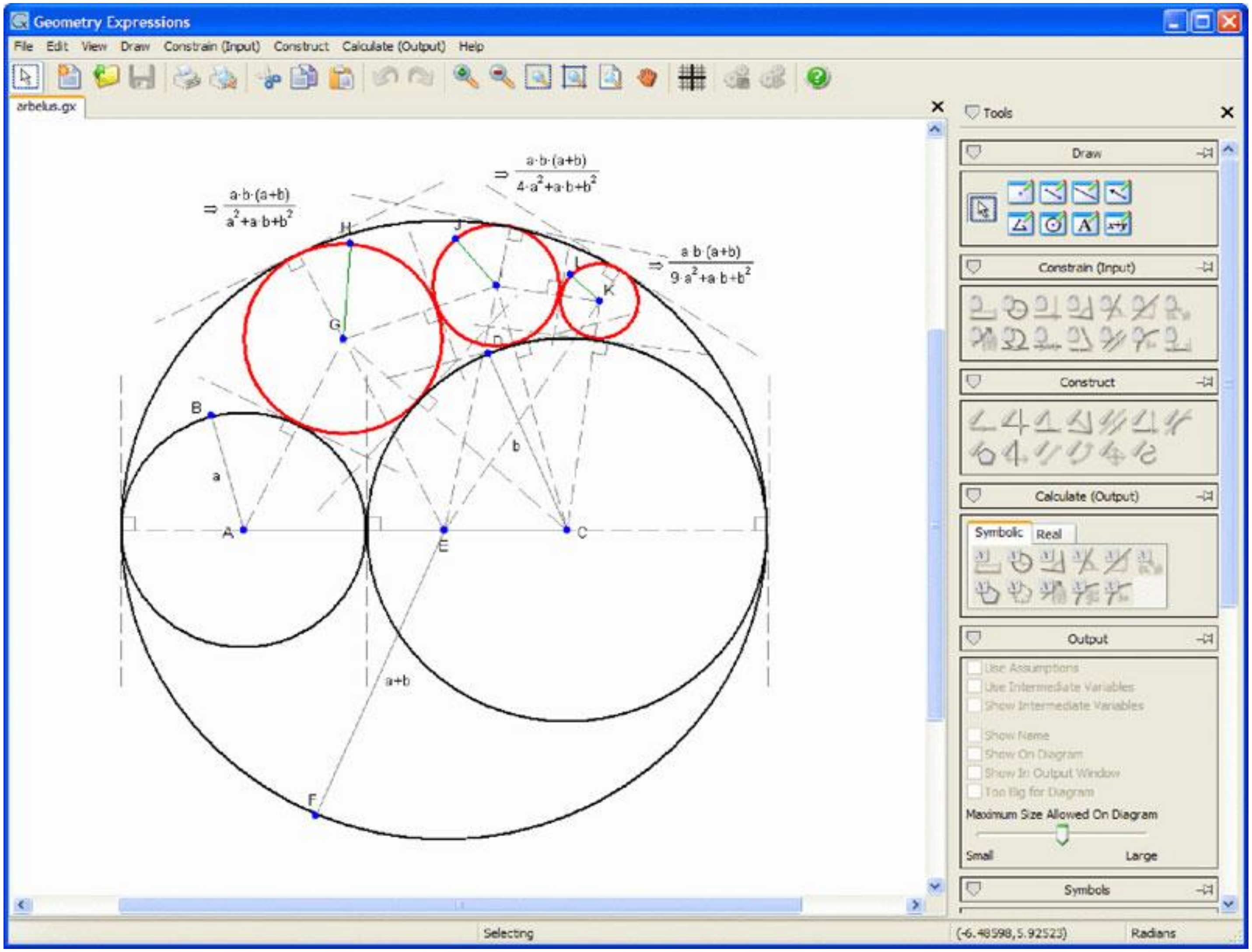


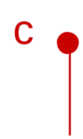
Figure 7-2: Rotation of the vector **x** through the rotor **ba**. The bivector part of the rotor (strong yellow) represents the plane of rotation, and the angle between the two vectors is half the rotation angle.

Symbolic Geometry



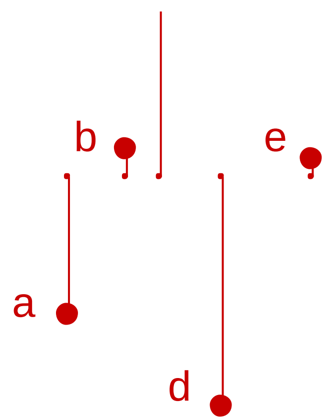
Lecture12-PolynomialCartDeriv.gx

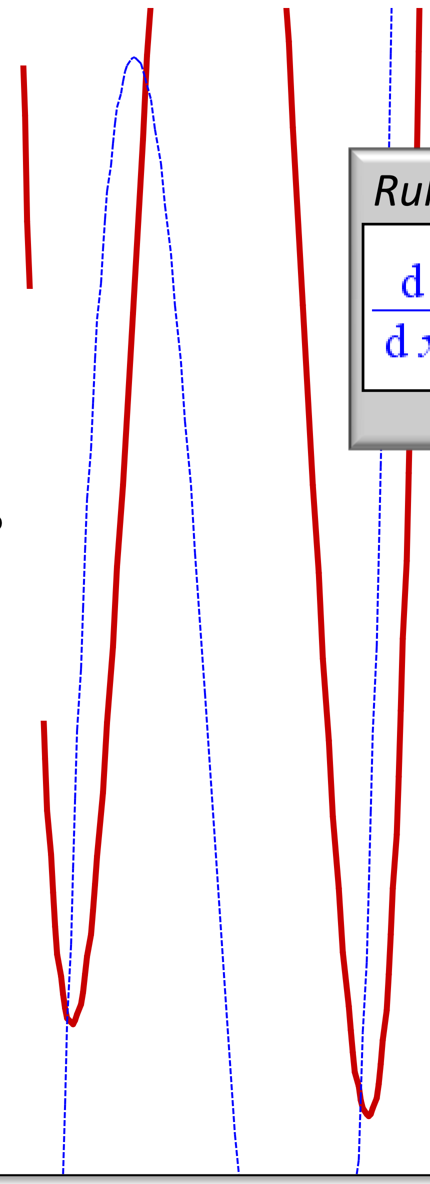
Polynomial Derivatives:



EXERCISES:

1) Drag a, b, c, d and e to produce x-axis crossings of 0, 2 and 4 roots.
2) When the derivative is zero, what happens to the original polynomial?
3) Take turns setting a – e to zero to reproduce lower order polynomials.
4) What happens to the derivative?





Rule:

$$\frac{d}{dx} k\cdot x^{n} = k\cdot n\cdot x^{n-1}$$

$$Y=a+X\cdot b+X^2\cdot c+X^3\cdot d+X^4\cdot e$$

$$Y'=b+2\cdot c\cdot X+3\cdot d\cdot X^2+4\cdot e\cdot X^3$$

Polar Derivatives Of Polar Functions

To find the derivative $dr/d\theta$ of a curve $r(\theta)$ we differentiate the curve in the usual fashion:

1) Consider a polynomial where the radius r is a function of the angle θ:

$$r(\theta) = a\,\theta^0 + b\,\theta^1 + c\theta^2 + d\theta^3 + e\theta^4$$

2) We rewrite this to obtain the familiar looking form:

$$r(\theta) = a + b\,\theta + c\theta^2 + d\theta^3 + e\theta^4$$

3) And differentiate in the usual fashion to obtain:

$$r'(\theta) = 0 + b + 2c\theta^1 + 3d\theta^2 + 4e\theta^3$$

$$r'(\theta) = b + 2c\theta + 3d\theta^2 + 4e\theta^3$$

Lecture12-PolynomialPolarDeriv.gx

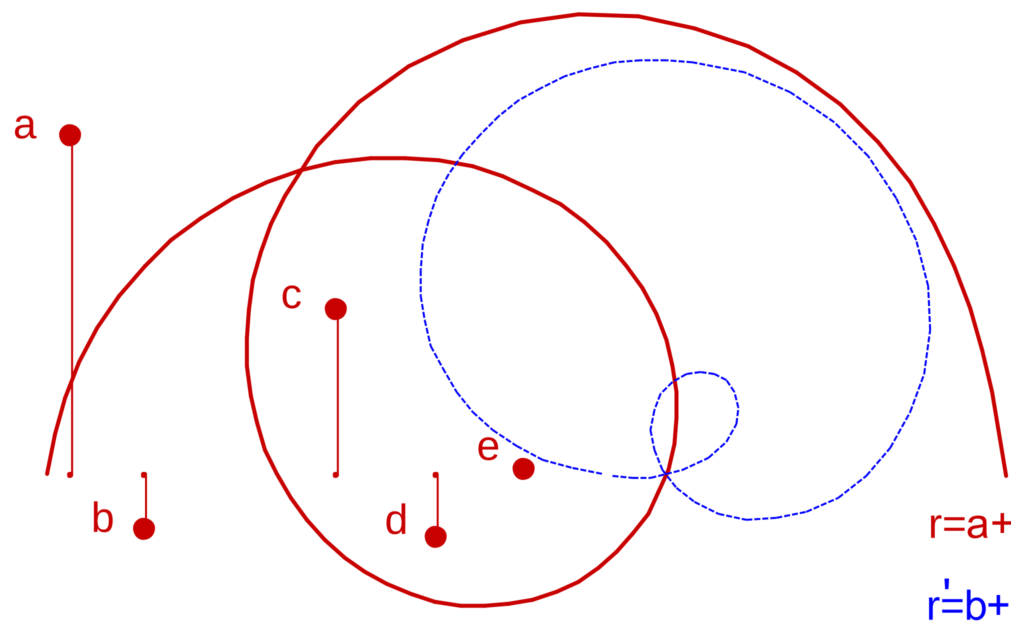
Polynomial Derivatives:

EXERCISES:

1) Take turns setting a – e to zero to reproduce lower order polynomials.
2) What happens to the derivative?
3) Write an expression for the slope of the line tangent to the original curve.

Rule:

$$\frac{d}{dx} k \cdot \theta^{n} = k \cdot n \cdot \theta^{n-1}$$



$r = a + T \cdot b + T^2 \cdot c + T^3 \cdot d + T^4 \cdot e$

$r' = b + 2 \cdot T \cdot c + 3 \cdot T^2 \cdot d + 4 \cdot T^3 \cdot e$

Approximation of Functions

It is often convenient to approximate one function using another function. When we are approximating a function in a neighborhood of the origin, the Maclaurin series is often used:

$$f(x) \approx f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \ldots$$

The series goes on, but when we are approximating a function with a polynomial of degree n, we know that there are only n non-zero derivatives, so we neglect the higher order terms.

Let's approximate the function $a + \sin(x)$ using the Maclaurin series:

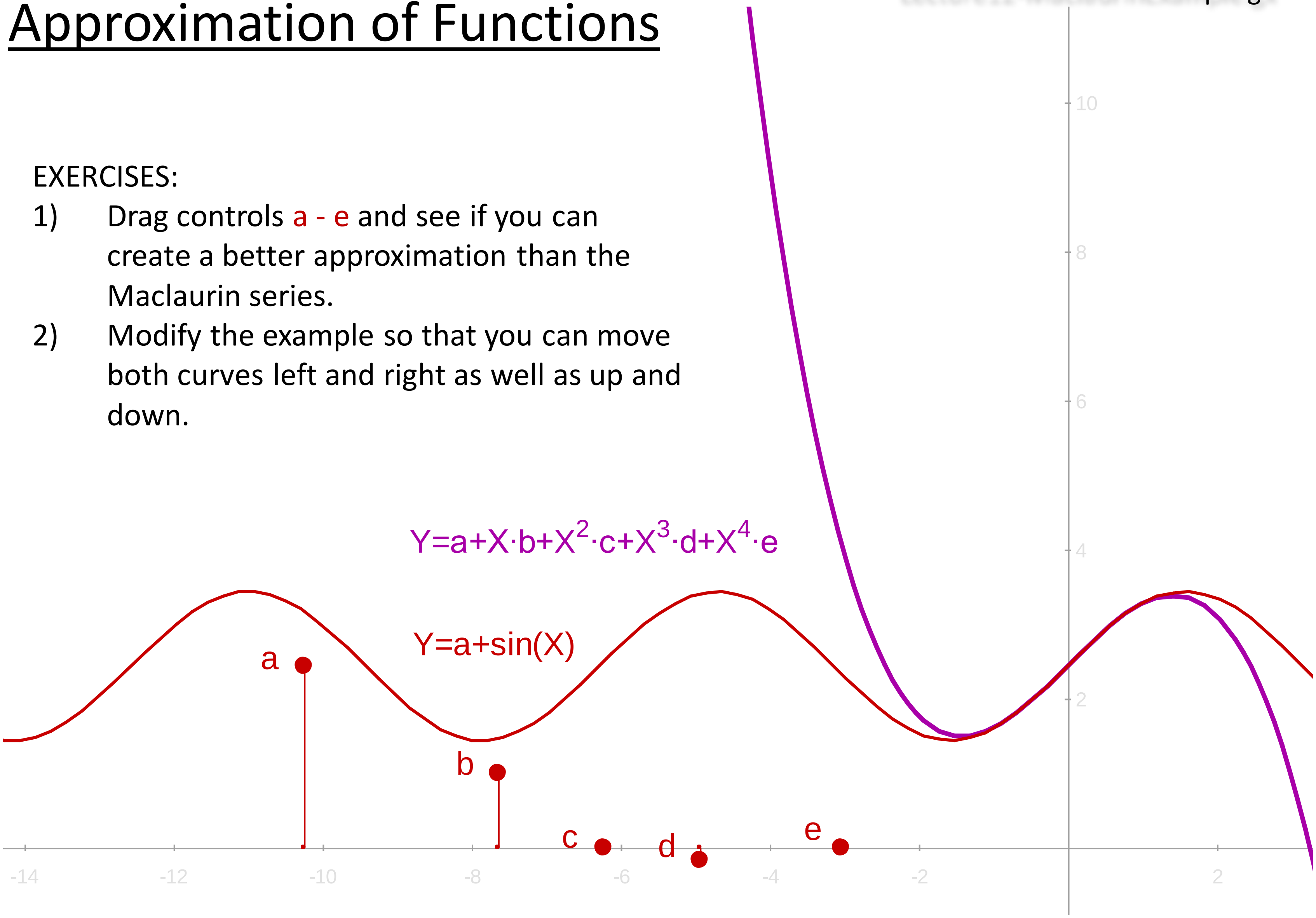
$$\begin{aligned} a + \sin(x) &\approx a + \sin(0) + \cos(0) \cdot x + \frac{-\sin(0)}{2!} \cdot x^2 + \frac{-\cos(0)}{3!} \cdot x^3 + \ldots \\ &\approx a + 0 \qquad x + 0 + \frac{-1}{6} \cdot x^3 + \ldots \end{aligned}$$

Approximation of Functions

Lecture12-MaclaurinExample.gx

EXERCISES:

1) Drag controls a - e and see if you can create a better approximation than the Maclaurin series.
2) Modify the example so that you can move both curves left and right as well as up and down.

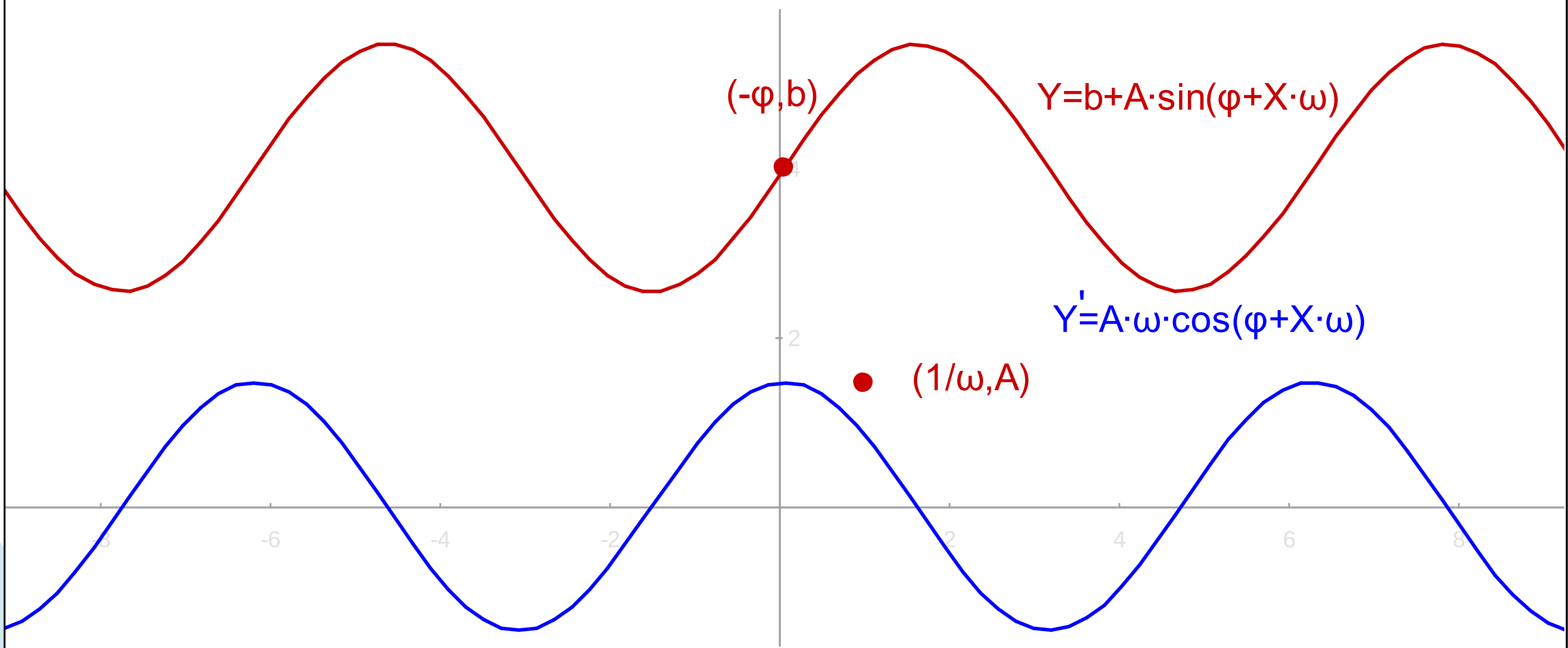


Derivative of Sine:

Lecture12-FullSinDeriv.gx

EXERCISES:

1) What do the parameters A, b, ω and ϕ control in the sine function?
2) Drag the control point for (-ϕ, b).
3) Why doesn't the derivative curve move up and down?.
4) Why does -ϕ make a better control than ϕ?
5) Drag the control point for (1/ω, A). Why does 1/ω make a better control than ω?
6) Revise the example to demonstrate the cosine and its derivative.

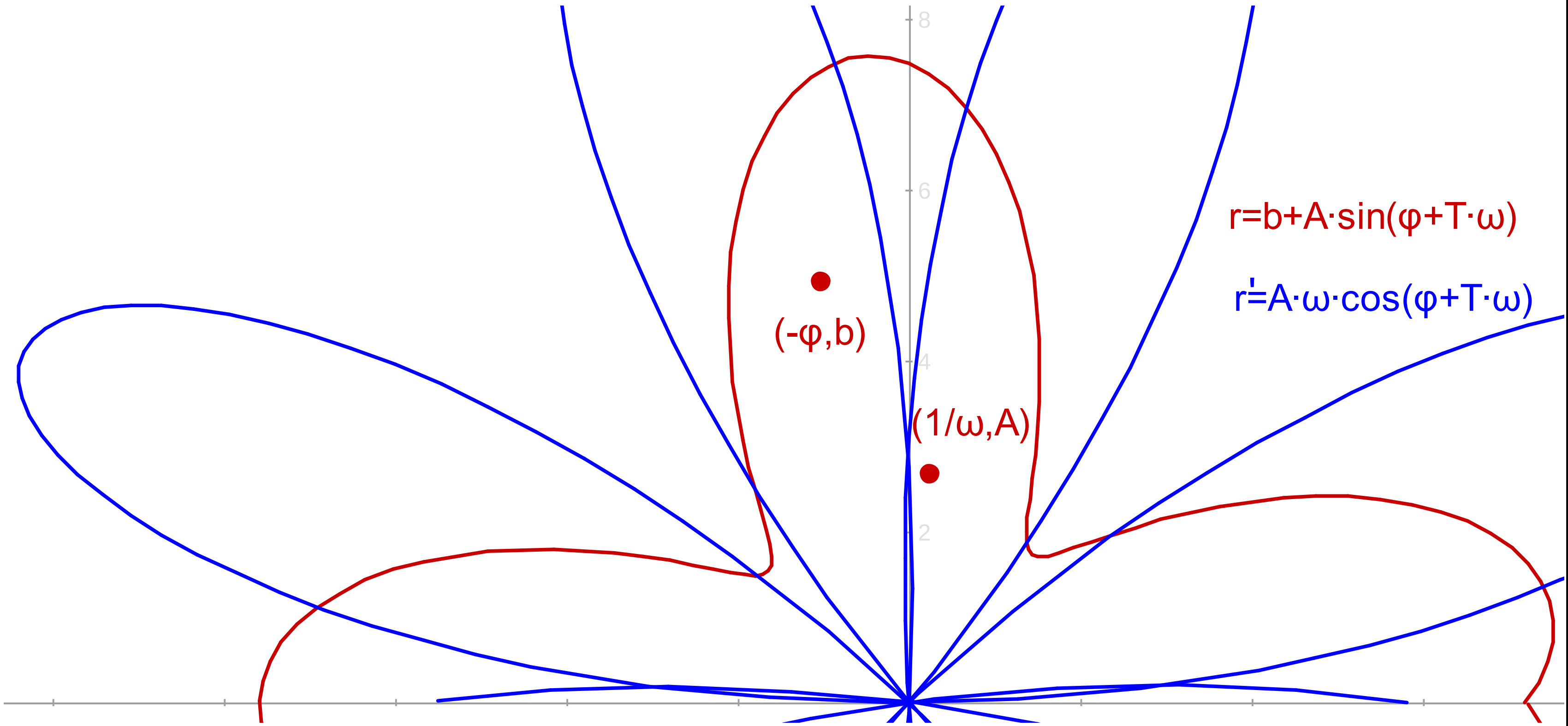


Derivative of Sine In Polar Coordinates:

Lecture12-FullSinDerivPolar.gx

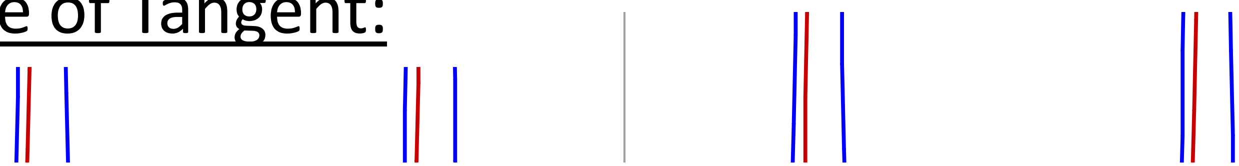
EXERCISES:

1) What do the parameters A, b, ω and ϕ control in the polar sine function?
2) Drag the control point for (-ϕ, b).
3) Drag the control point for (1/ω, A).
4) Revise the example for cosine and its derivative in polar coordinates.



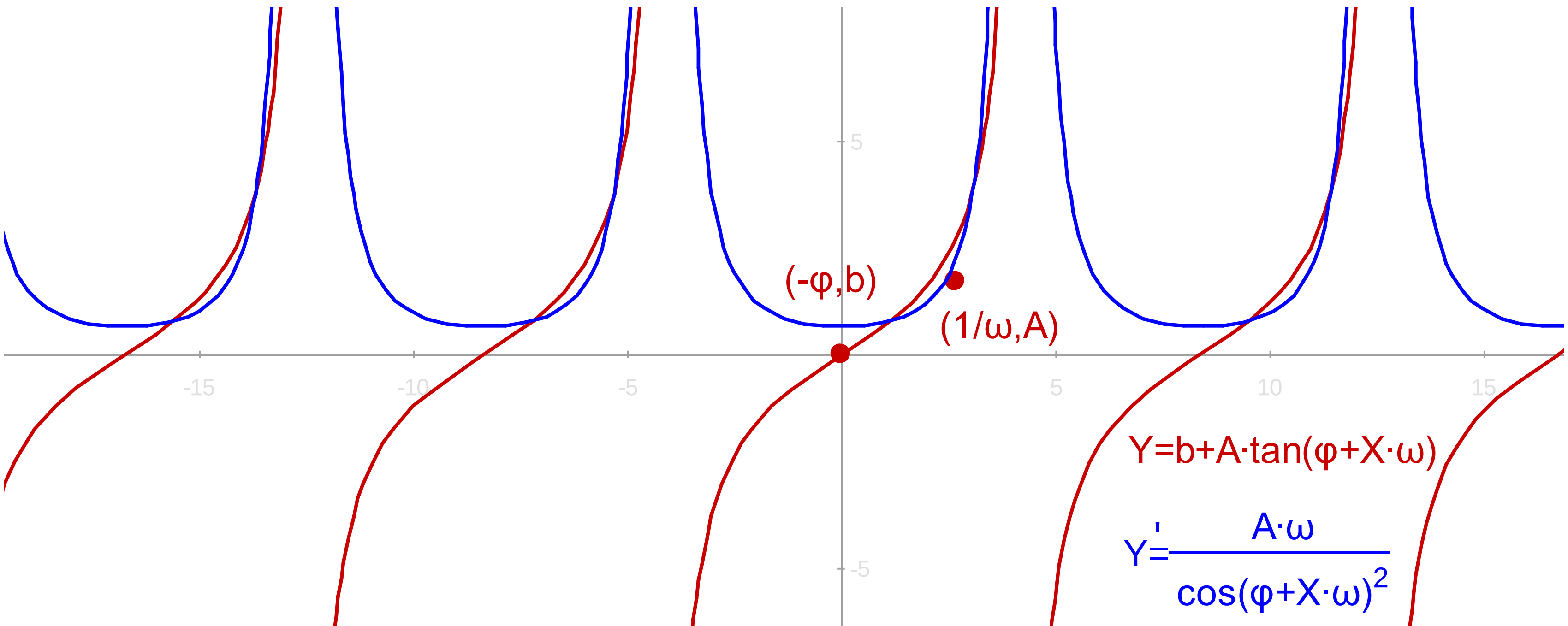
Derivative of Tangent:

Lecture12-FullTanDeriv.gx



EXERCISES:

1) What do the parameters A, b, ω and ϕ control in the tangent function?
2) Why doesn't the derivative curve move up and down when b is changed?
3) Drag the controls for (-ϕ, b) and (1/ω, A).
4) Revise the example to demonstrate the cotangent and derivative.
5) For what values does the function make a good approximation to its derivative?

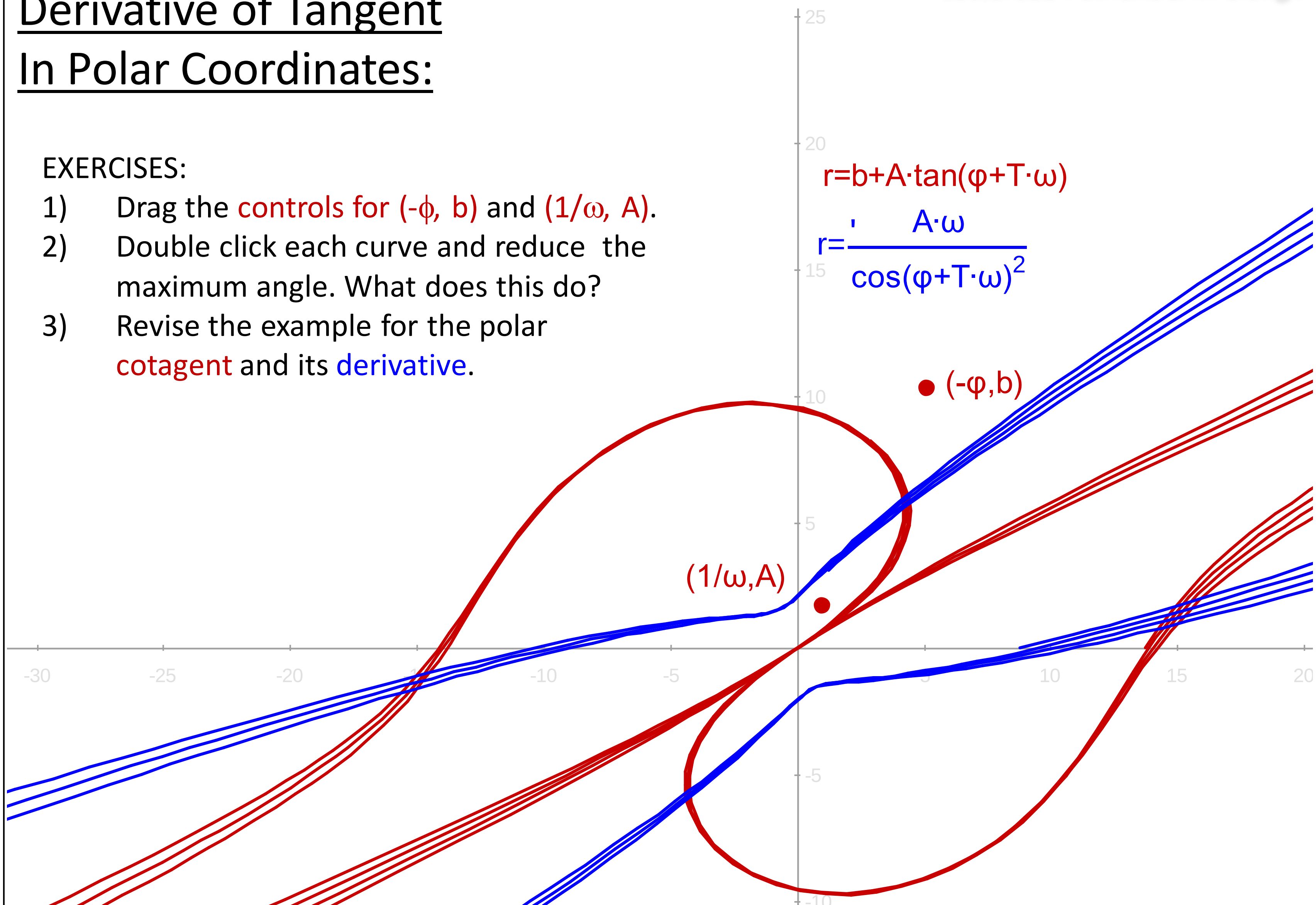


Lecture12-FullTanDerivPolar.gx

Derivative of Tangent In Polar Coordinates:

EXERCISES:

1) Drag the controls for (-ϕ, b) and (1/ω, A).
2) Double click each curve and reduce the maximum angle. What does this do?
3) Revise the example for the polar cotagent and its derivative.

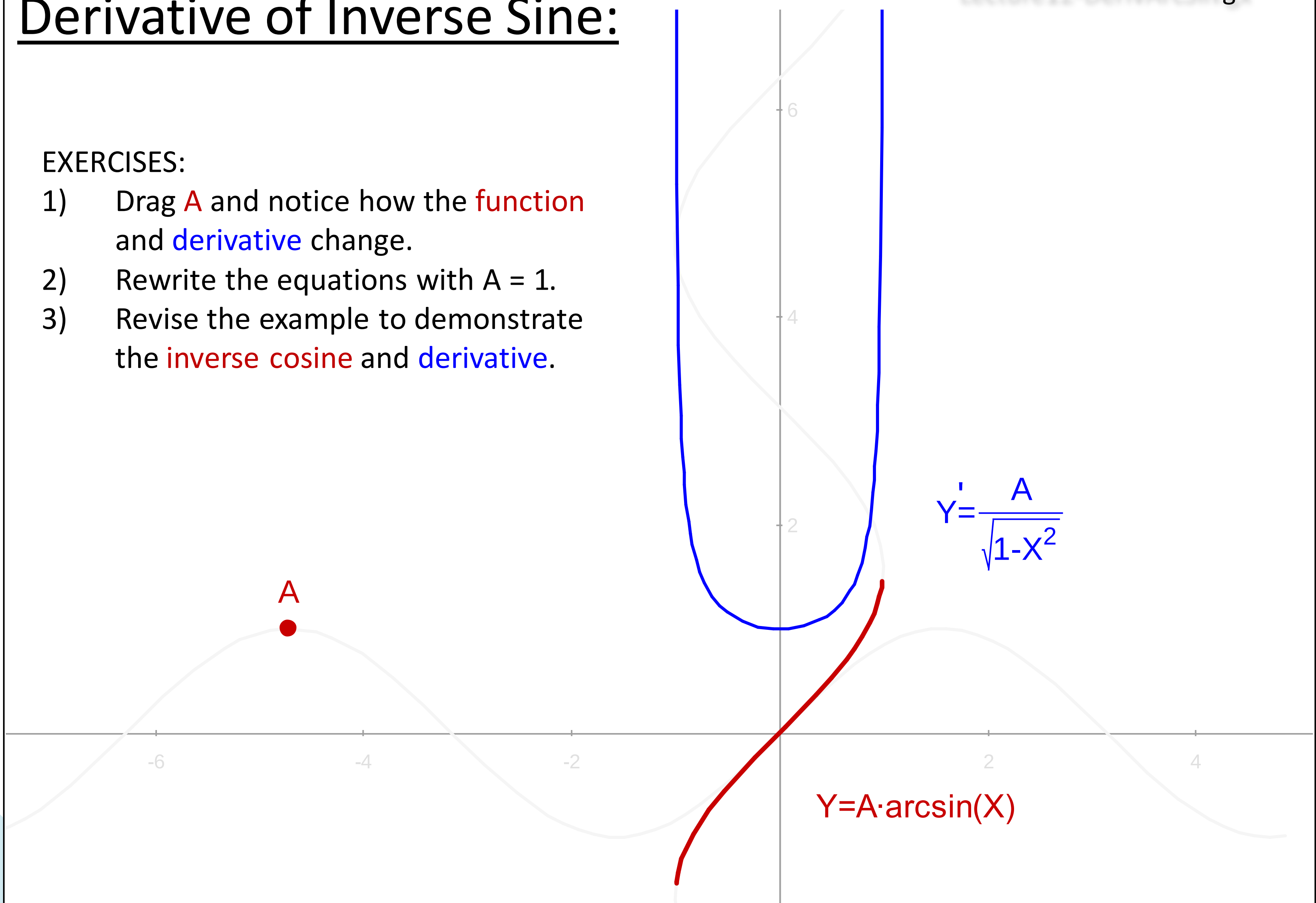


Derivative of Inverse Sine:

Lecture12-DerivArcSin.gx

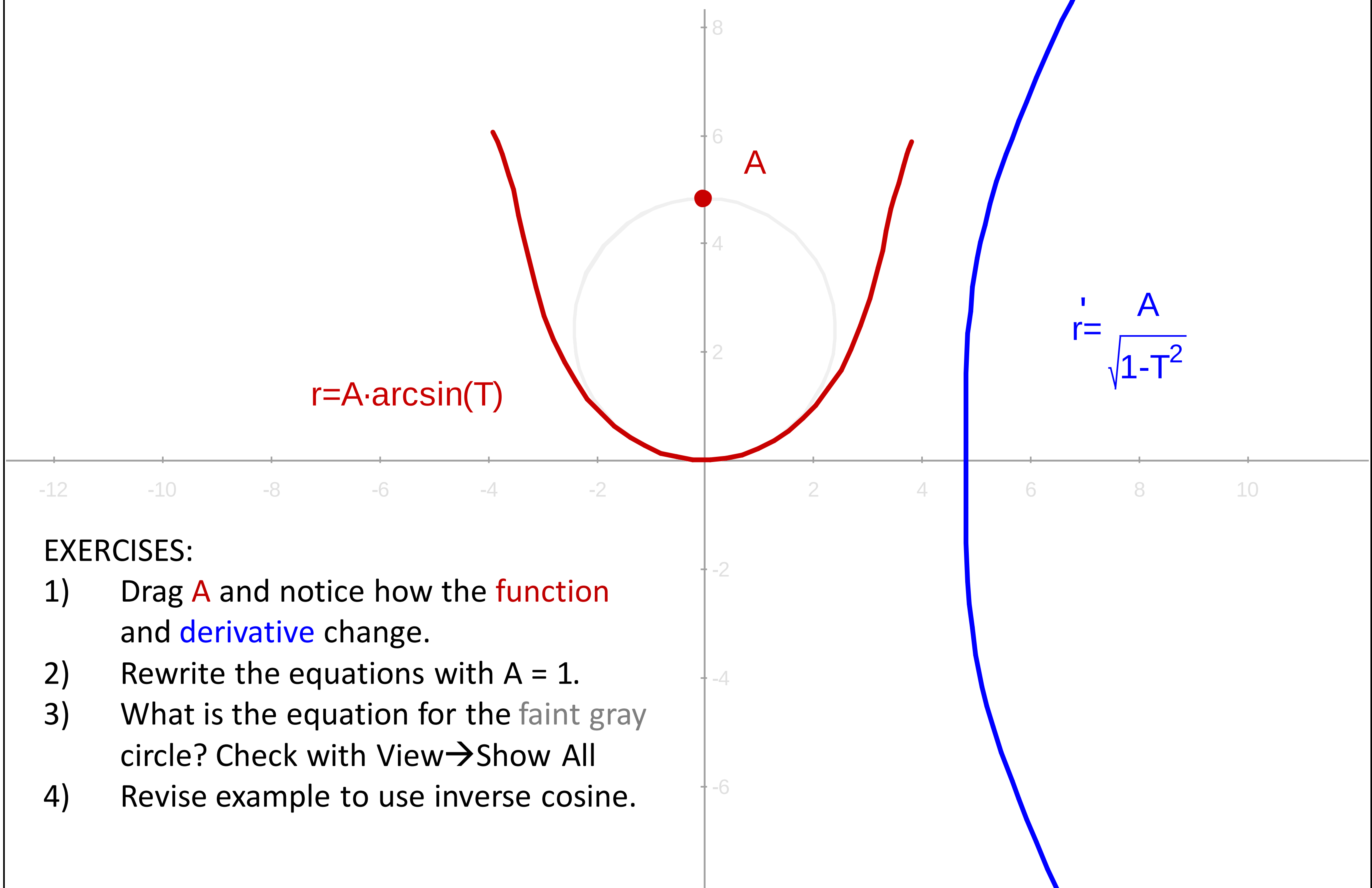
EXERCISES:

1) Drag A and notice how the function and derivative change.
2) Rewrite the equations with A = 1.
3) Revise the example to demonstrate the inverse cosine and derivative.



Lecture12-DerivArcSinPolar.gx

Derivative of Polar Inverse Sine:



EXERCISES:

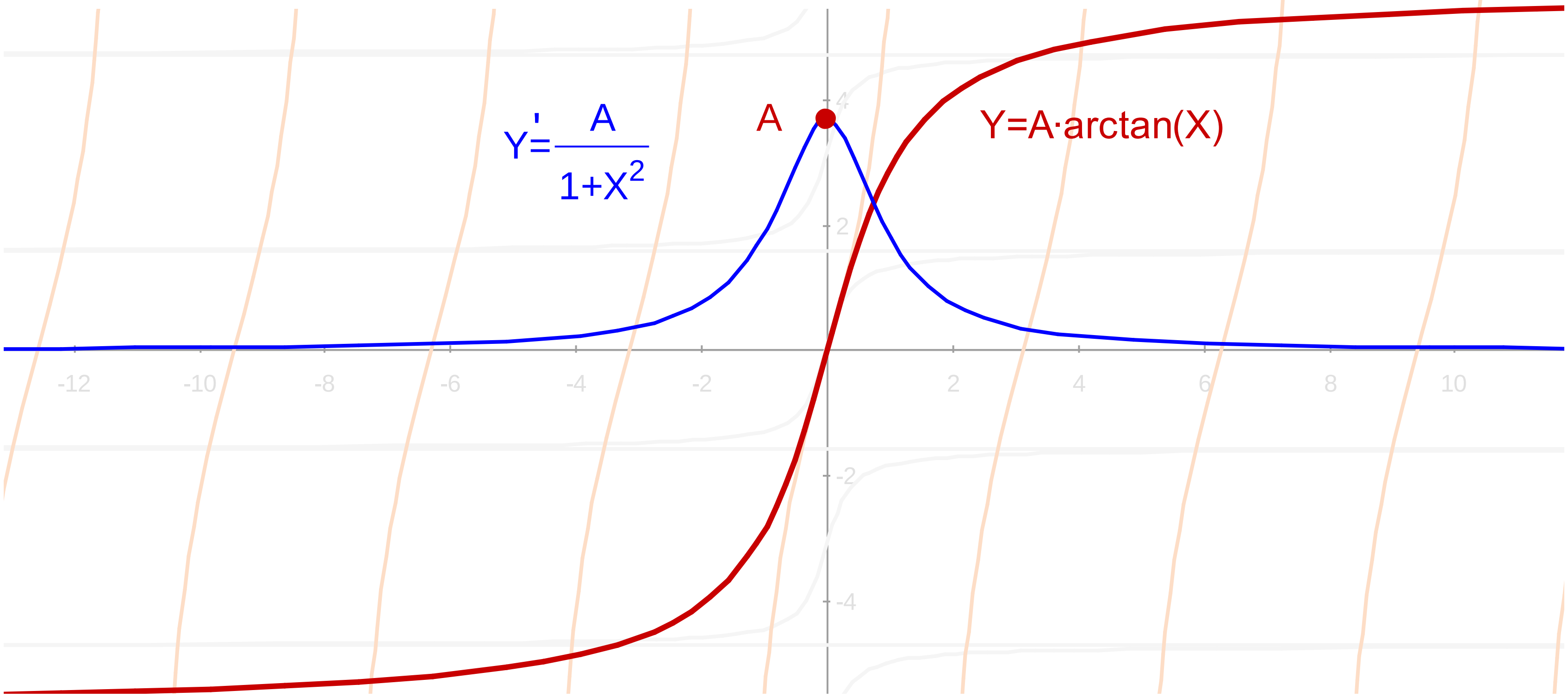
1) Drag A and notice how the function and derivative change.
2) Rewrite the equations with A = 1.
3) What is the equation for the faint gray circle? Check with View→Show All
4) Revise example to use inverse cosine.

Derivative of Inverse Tangent:

Lecture12-DerivArcTan.gx

EXERCISES:

1) Drag A and notice how the space formed by the grid is warped.
2) How would you address a coordinate system in this space?
3) Rewrite the equations with A = 1.
4) What are the faint horizontal gray lines?
5) What are the faint vertical orange lines?

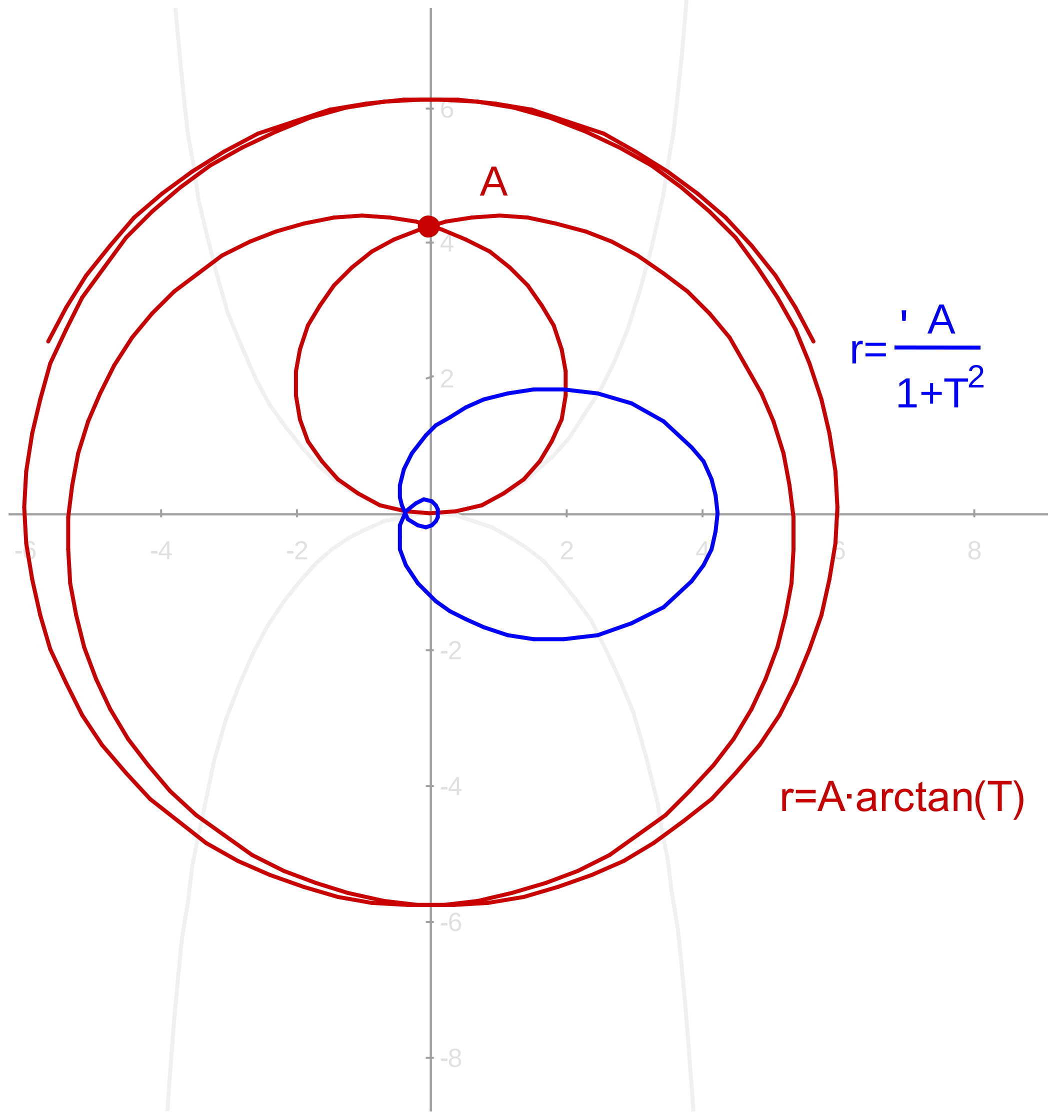


Lecture12-DerivArcTanPolar.gx

Derivative of Polar Inverse Tangent:

EXERCISES:

1) Drag A and notice how the function and derivative change.
2) Rewrite the equations with A = 1.
3) What is the equation for the faint gray circle? Check your work with View→Show All
4) Double-click the function and change its start and end angle to -20 and +20, respectively.
   What happens?



Sinc and Cinc Function:

Lecture12-SincFunction.gx

EXERCISES:

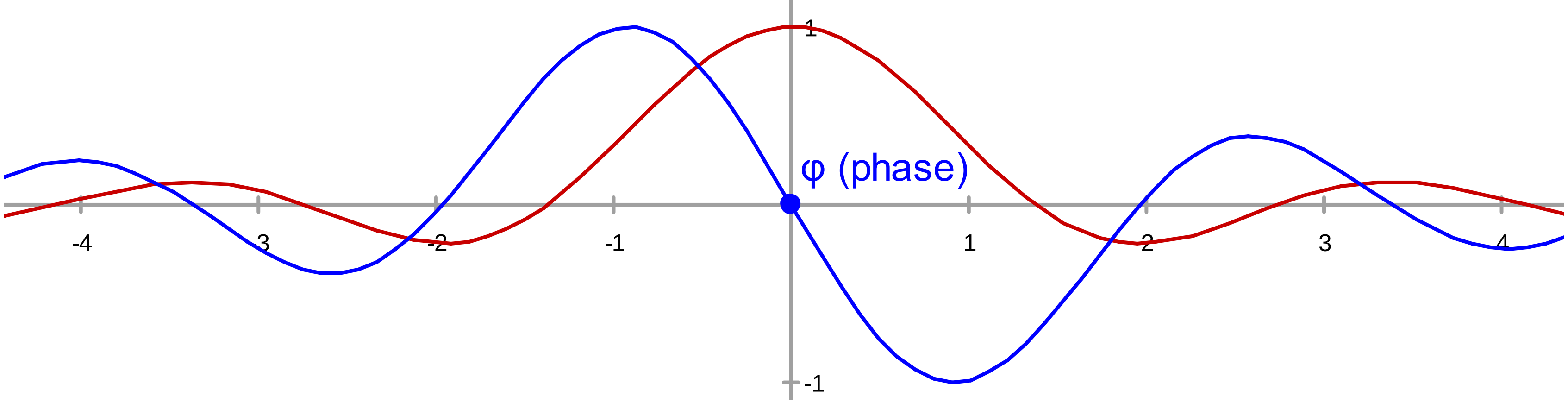
1) Drag a to find a value at which the sinc function and its derivative are close to the same maximum height.
2) What is the exact value?
3) Drag the phase ϕ left and right. Can you get *any* portion of the two curves to match? How would you evaluate the quality of the match?
4) The blue curve is the derivative of the red curve at only one value of phase. What is that value?

Cartesian Form:

$$y(x) = \frac{\sin(a\,x)}{a\,x}$$

$$\frac{d}{dx}\,y(x) = \frac{\cos(a\,x)}{x} - \frac{\sin(a\,x)}{a\,x^2}$$

a (amplitude)



Cartesian Derivatives Of Polar Functions

To find the slope dy/dx of the line tangent to a curve $r(\theta)$ we follow a three step process:

1) Use the Cartesian coordinate system conversion:

$$x = r(\theta)\cos\theta$$

$$y = r(\theta)\sin\theta$$

2) Differentiate both equations against θ:

$$dx/d\theta = r'(\theta)\cos\theta - r(\theta)\sin\theta$$

$$dy/d\theta = r'(\theta)\sin\theta + r(\theta)\cos\theta$$

3) Divide the second equation by the first to obtain $dy/dx =$

$$\frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

Lecture12-CochleoidTangent.wxm

Cartesian Derivative Of Cochleoid

Here we use wxMaxima™ to find the slope of the cochleoid – the polar sinc function:

```
(%i1) r(t):=sin(a*t)/(a*t)$

(%i2) x(t):=r(t)*cos(t)$

(%i3) y(t):=r(t)*sin(t)$

(%i4) tangent(t):=diff(y(t),t)/diff(x(t),t)$

(%i5) radcan(tangent(t));
```

(%o5) $$\frac{(\sin(t)-t\cos(t))\sin(a\,t)-a\,t\sin(t)\cos(a\,t)}{(t\sin(t)+\cos(t))\sin(a\,t)-a\,t\cos(t)\cos(a\,t)}$$

Note that $ at the end of a line suppresses output, and that radcan simplifies the expression.

Cochleoid Function:

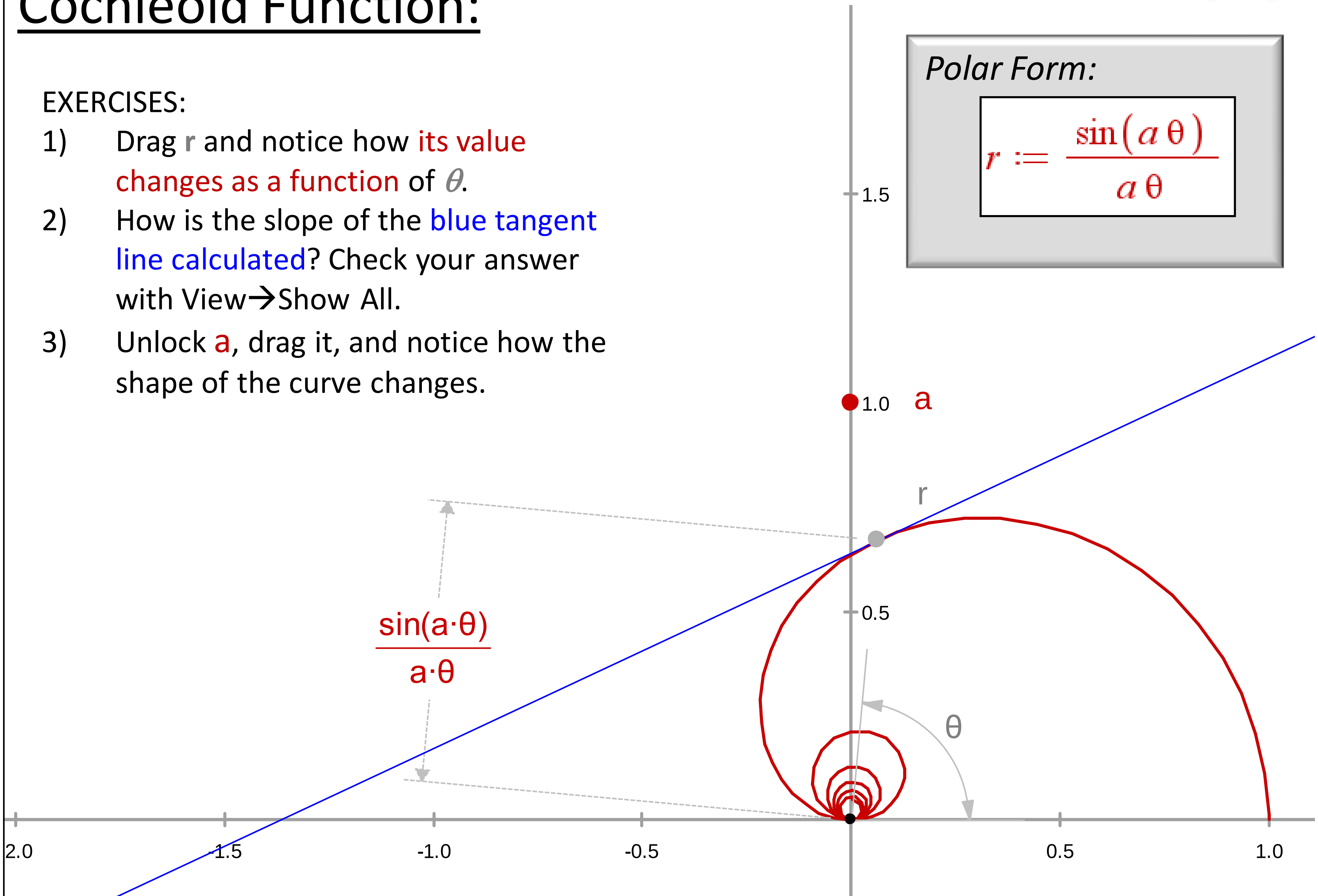
Lecture12-CochleoidTangent.gx

EXERCISES:

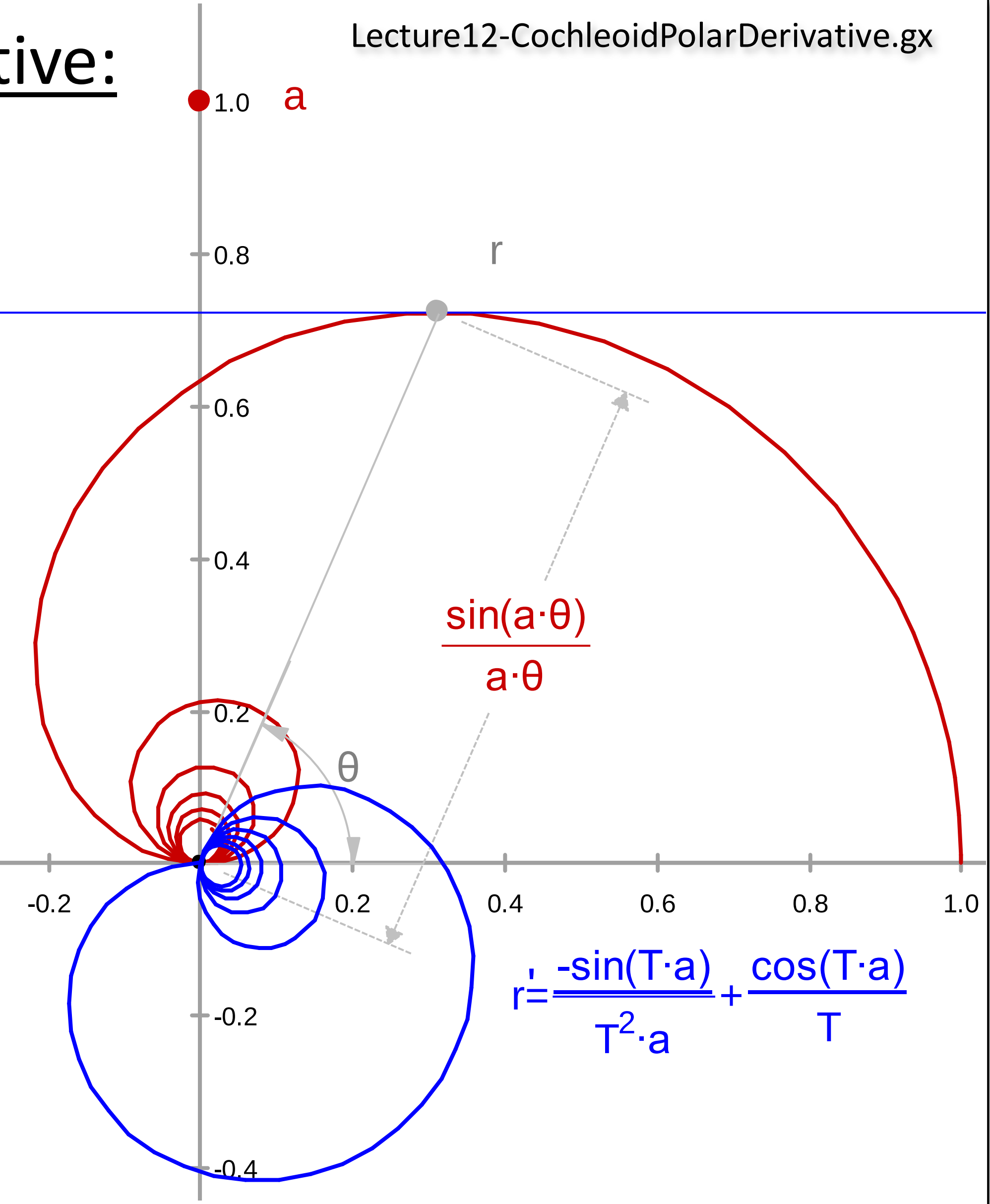
1) Drag r and notice how its value changes as a function of θ.
2) How is the slope of the blue tangent line calculated? Check your answer with View→Show All.
3) Unlock a, drag it, and notice how the shape of the curve changes.

Polar Form:

$$r := \frac{\sin(a\,\theta)}{a\,\theta}$$



Cochleoid Polar Derivative:

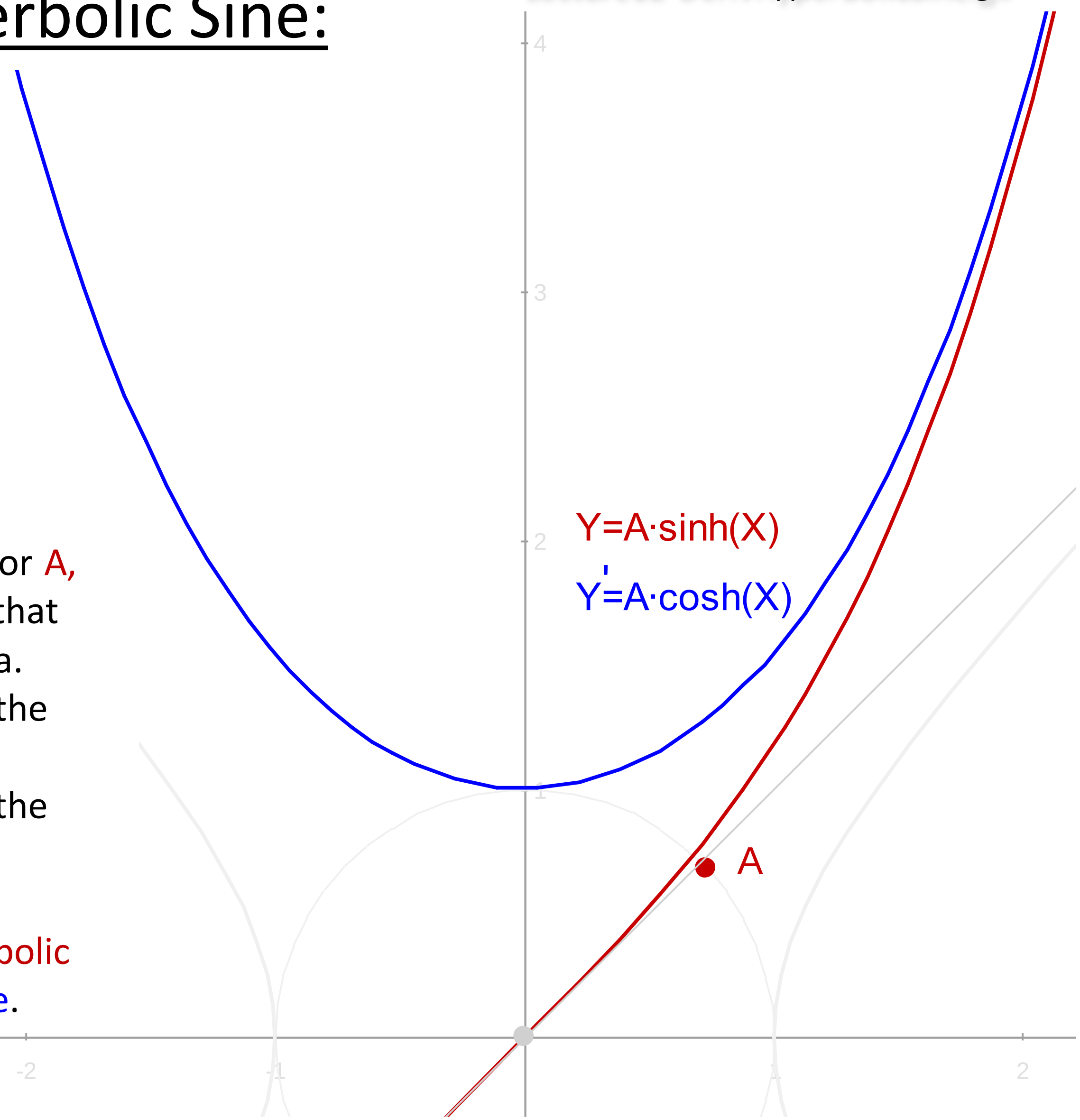


EXERCISES:

1) Unlock a, drag it, and notice how the shape of the original curve and its polar derivative changes.
2) dy/dx is the slope of the line tangent to the original curve. What is the meaning of dr/dθ, that is, r'?

Derivative of Hyperbolic Sine:

Lecture12-DerivHyperbolicSine.gx

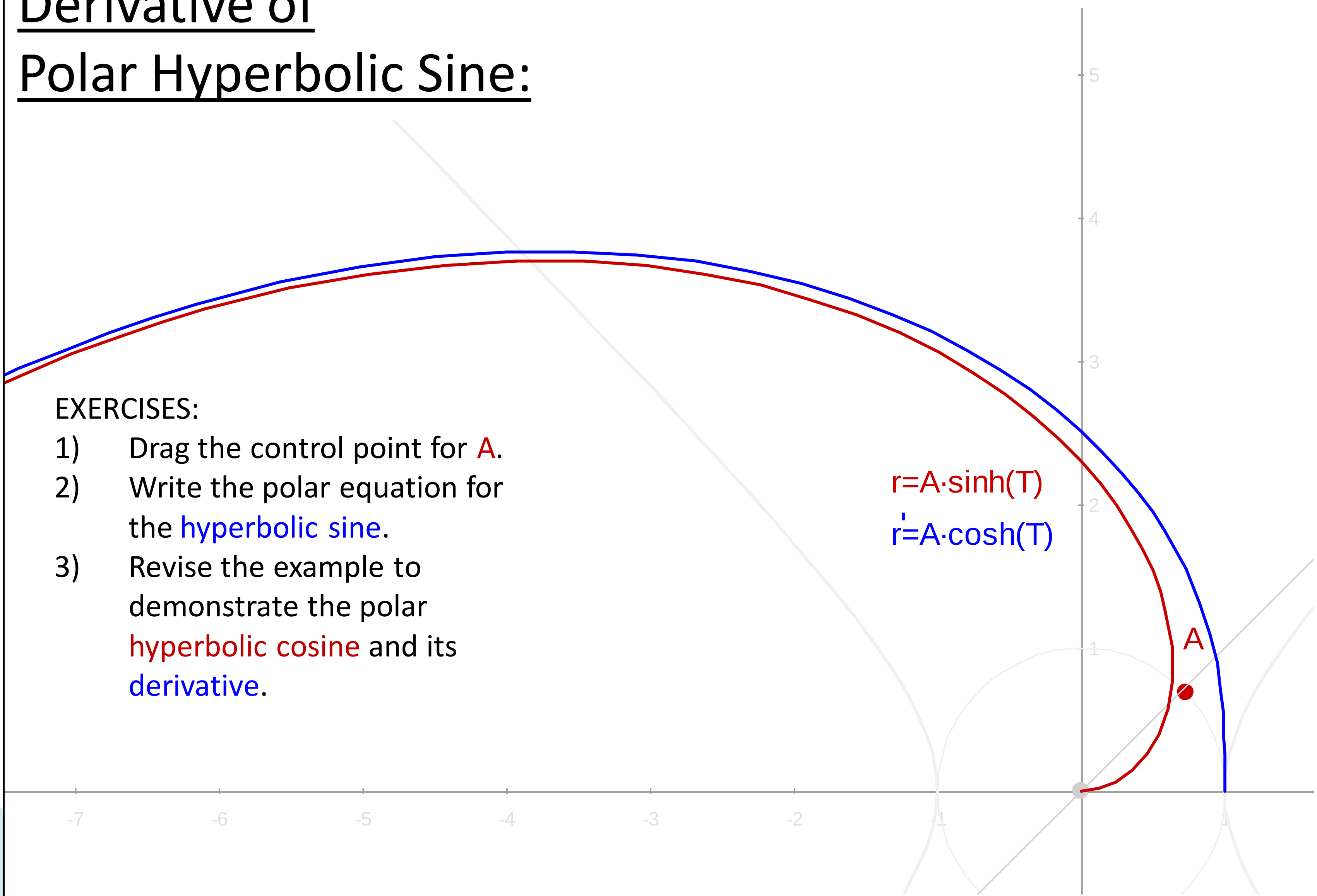


EXERCISES:

1) Drag the control point for A, the radius of the circle that sits inside the hyperbola.
2) Write the equation for the hyperbolic sine.
3) Write the equation for the bounding hyperbola.
4) Revise the example to demonstrate the hyperbolic cosine and its derivative.

Derivative of Polar Hyperbolic Sine:

Lecture12-DerivHyperbolicSinePolar.gx



EXERCISES:

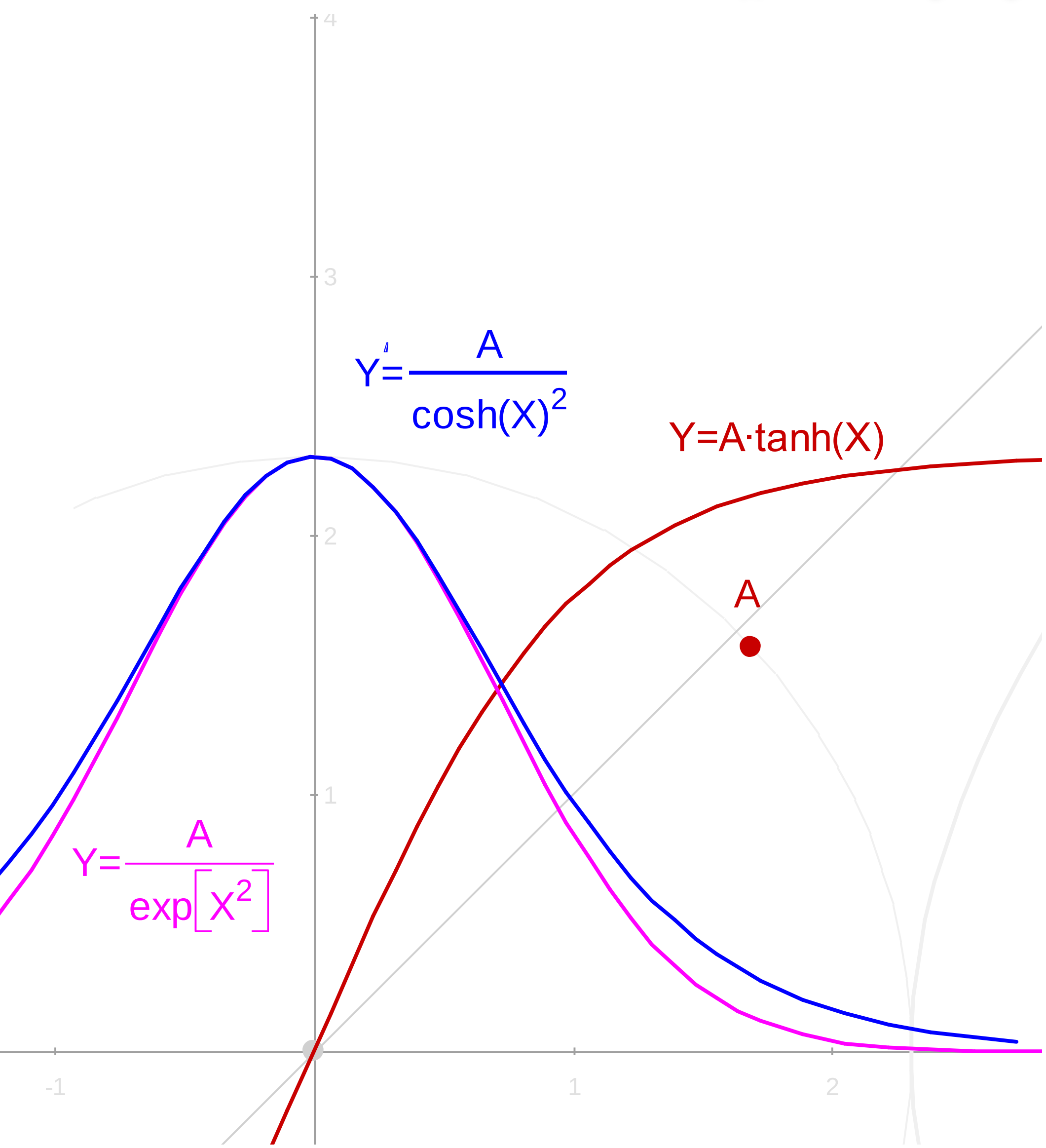
1) Drag the control point for A.
2) Write the polar equation for the hyperbolic sine.
3) Revise the example to demonstrate the polar hyperbolic cosine and its derivative.

Lecture12-DerivHyperbolicTangent.gx

Derivative of Hyperbolic Tangent:

EXERCISES:

1) Drag the control point for A.
2) Write the equation for the hyperbolic tangent, tanh(X).
3) How does one measure the closeness of tanh(X) as an approximation to the Gaussian curve shown? Which goes to zero fastest?

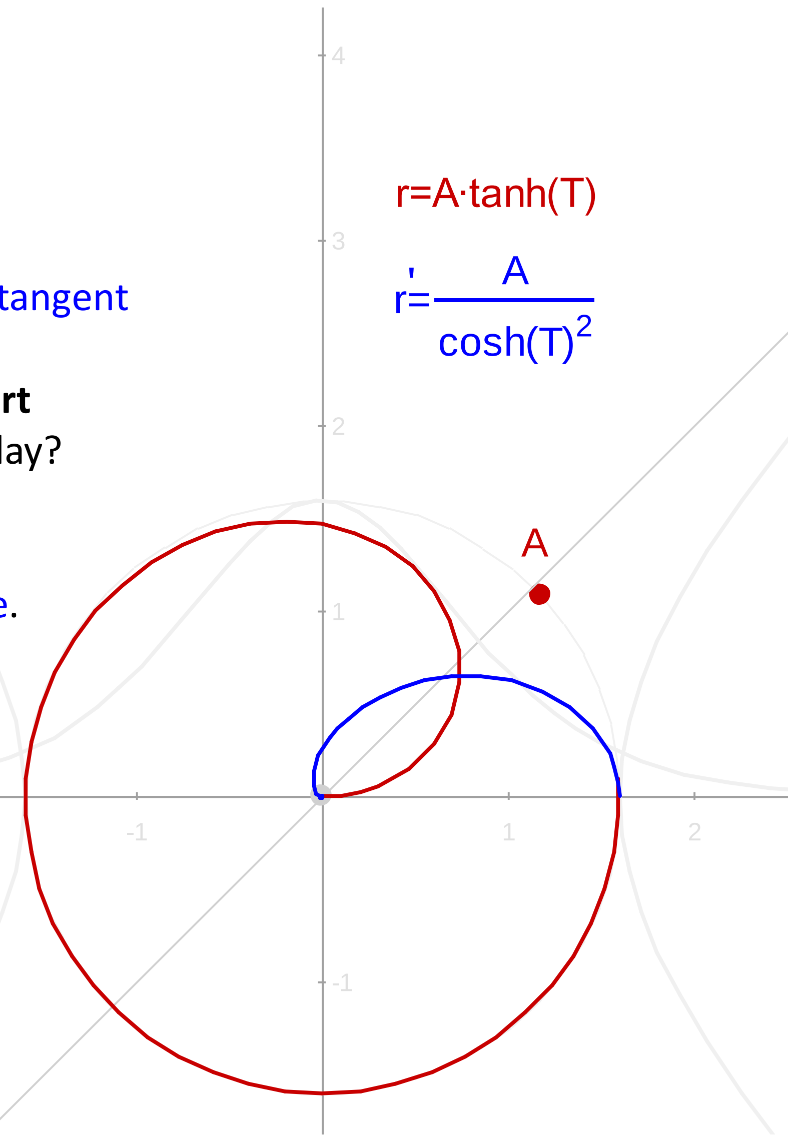


Derivative of Polar Hyperbolic Sine:

Lecture12-DerivHyperbolicTangentPolar.gx

EXERCISES:

1) Drag the control point for A.
2) Write the polar equation for the hyperbolic tangent in terms of exponentials.
3) Double-click on the function and change **Start** to -10. How does this change the curve display? Repeat with the derivative.
4) Revise the example to demonstrate the polar hyperbolic cotangent and its derivative.

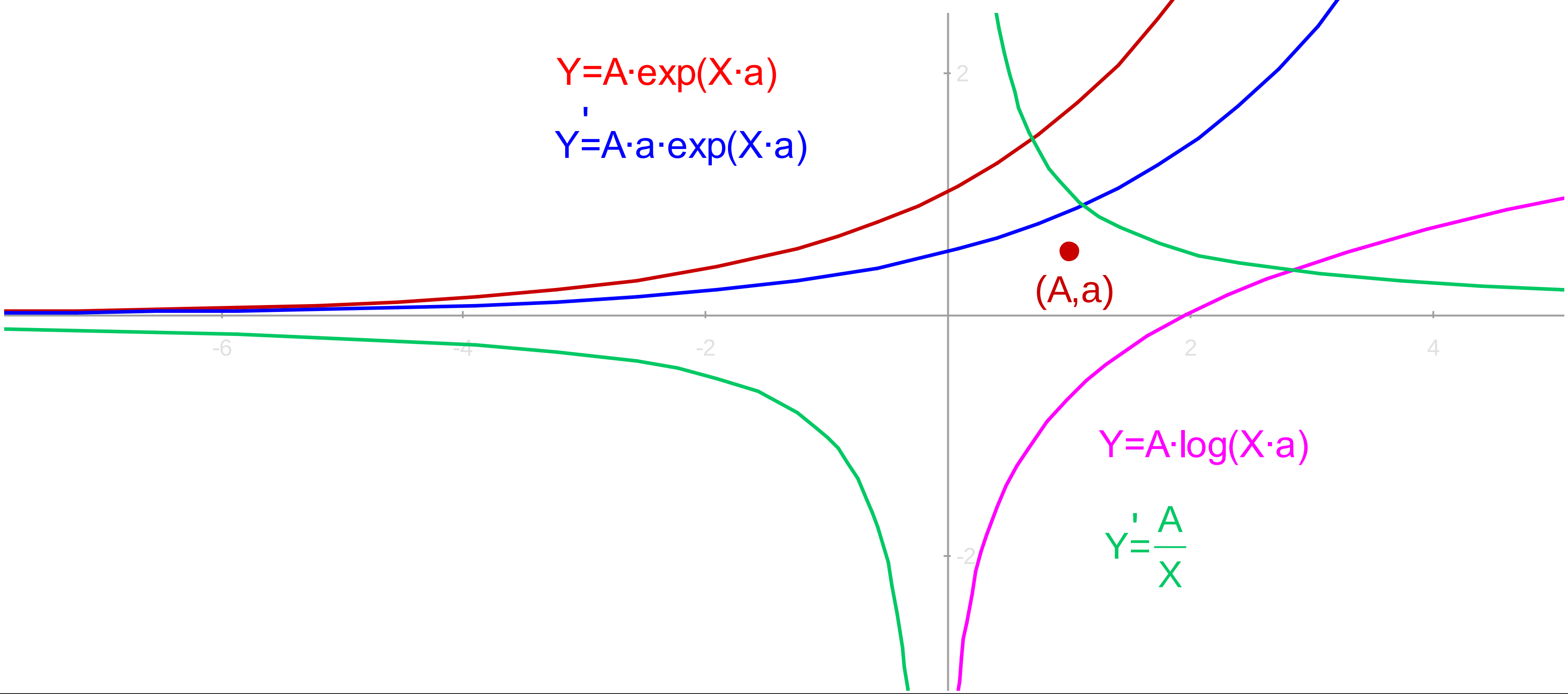


Lecture12-DerivLogExp.gx

Derivative of Exponential & Logarithm:

EXERCISES:

1) Drag the control point for (A, a) in a circle around the origin.
2) For what values of a are the exponential curve and its derivative curve identical?
3) Prove the exponential and logarithm curves are inverses.
4) Use the property of logarithms: (log(ab) = log(a) + log(b)) to show that log(2x) has the same as the derivative of log(x).

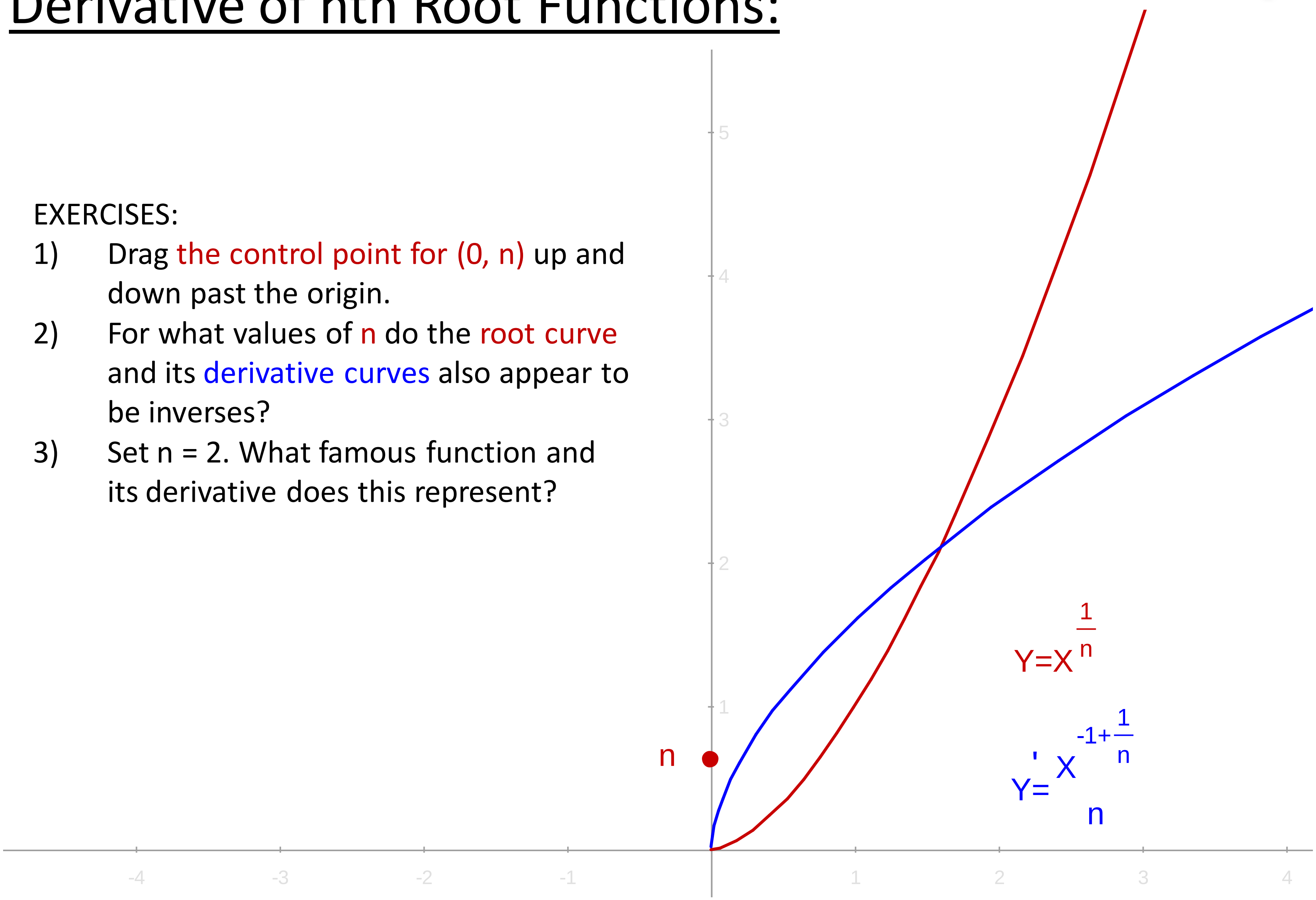


Derivative of nth Root Functions:

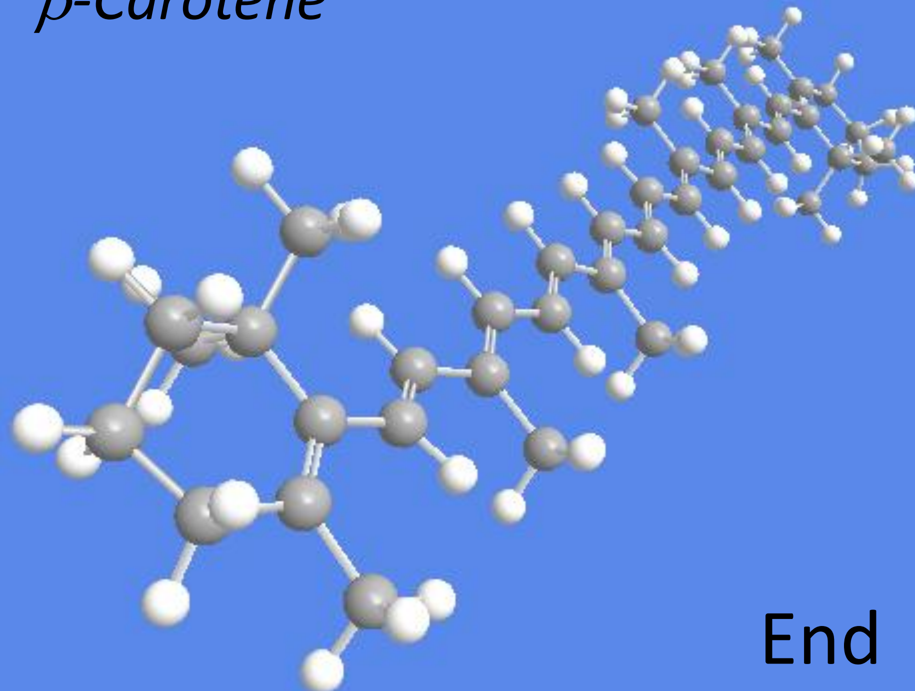
Lecture12-DerivRootFunctions.gx

EXERCISES:

1) Drag the control point for (0, n) up and down past the origin.
2) For what values of n do the root curve and its derivative curves also appear to be inverses?
3) Set n = 2. What famous function and its derivative does this represent?



β-Carotene



End