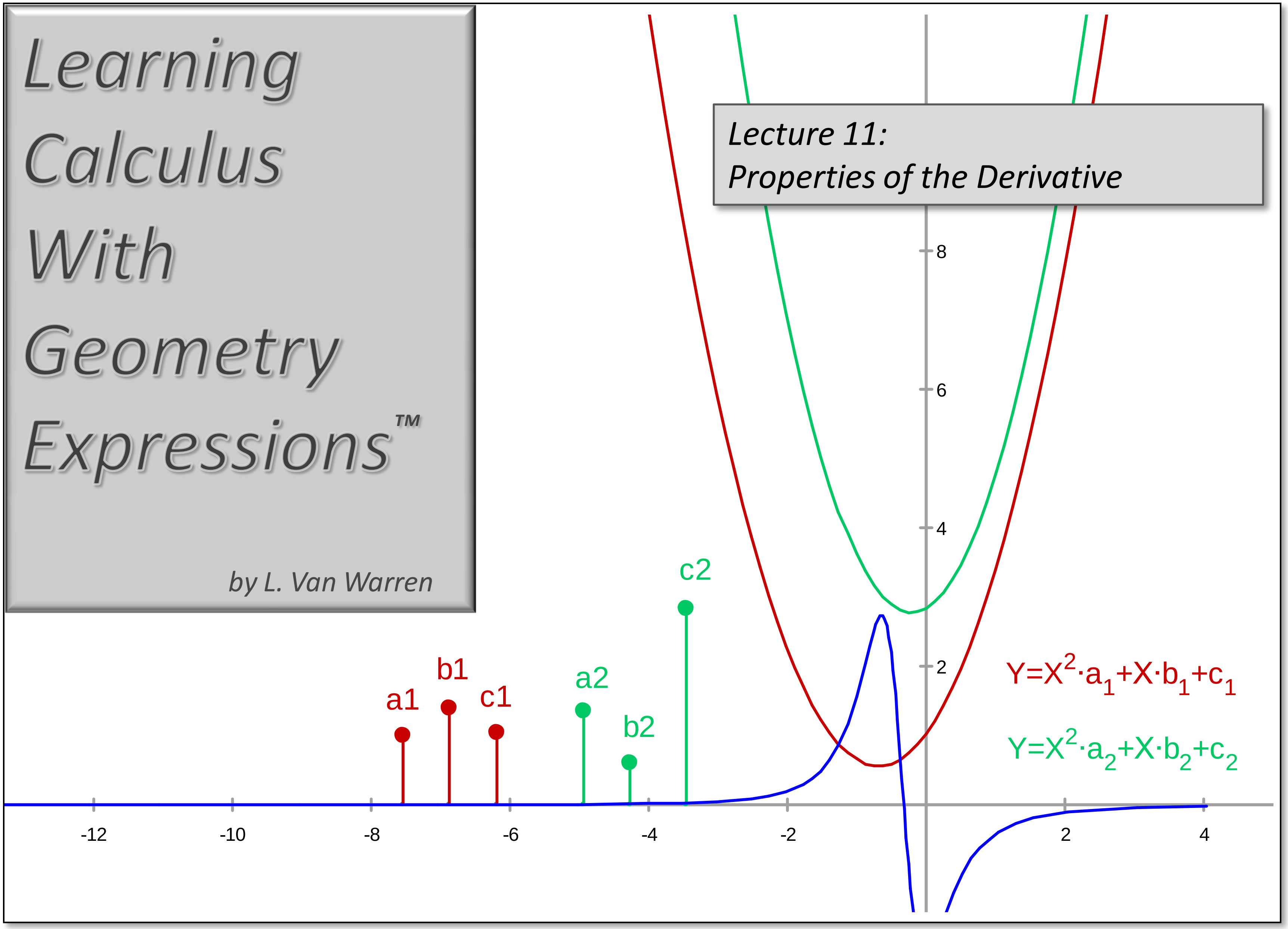
Learning Calculus With Geometry Expressions™

by L. Van Warren

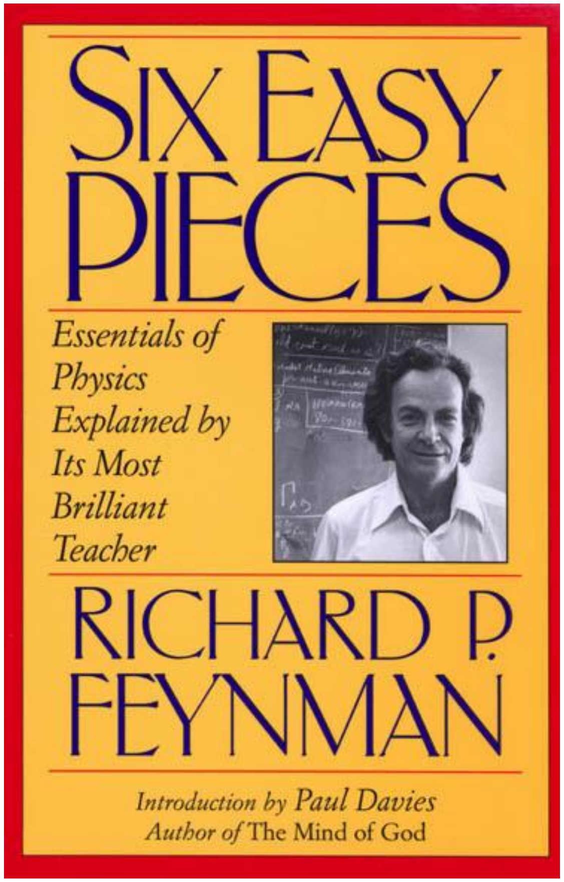
Lecture 11: Properties of the Derivative



Chapter 3: Derivatives

LECTURE	TOPIC
10	DEFINITION OF THE DERIVATIVE
11	**PROPERTIES OF THE DERIVATIVE**
12	DERIVATIVES OF COMMON FUNCTIONS
13	IMPLICIT DIFFERENTIATION

Inspiration



Richard Feynman
1918 - 1988

Quantum Electrodynamics - QED
Feynman Diagrams
Superfluidity of Helium 3

Won 1965 Nobel Prize

Noted for Clear Thinking and Presentation

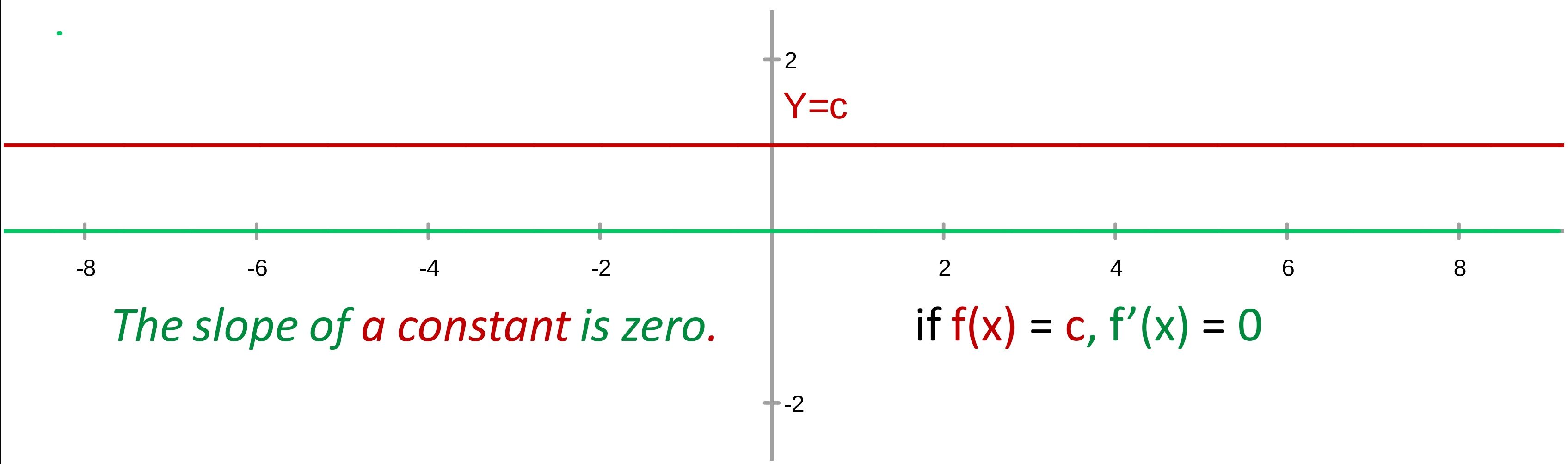
Recommended Reading

Properties of Derivatives

Lecture11-DerivativeOfAConstant.gx

The Derivative of a Constant is Zero!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$



The slope of a constant is zero.

if $f(x) = c$, $f'(x) = 0$

Properties of Derivatives

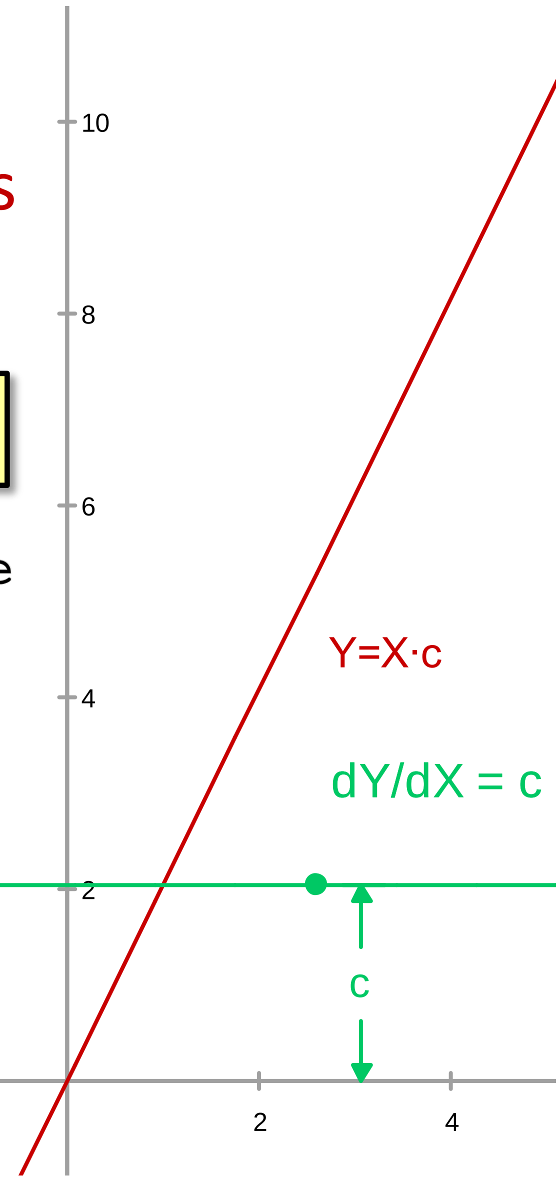
Lecture11-DerivativeConstTimesFunc.gx

Derivative of a Constant × Function is a Constant × Derivative

$$(c \cdot f)' = c \cdot f'$$

Proof:

$(g \cdot f)' =$	$g'f + f'g$	Product Rule
$(c \cdot f)' =$	$c'f + f'c$	Let $g = c$
$(c \cdot f)' =$	$0\ f + c\ f'$	Commut. Law
$(c \cdot f)' =$	$c\ f'$	QED



Properties of Derivatives

Lecture11-DerivativeOfALine.gx

The Derivative of a Line is its Slope!

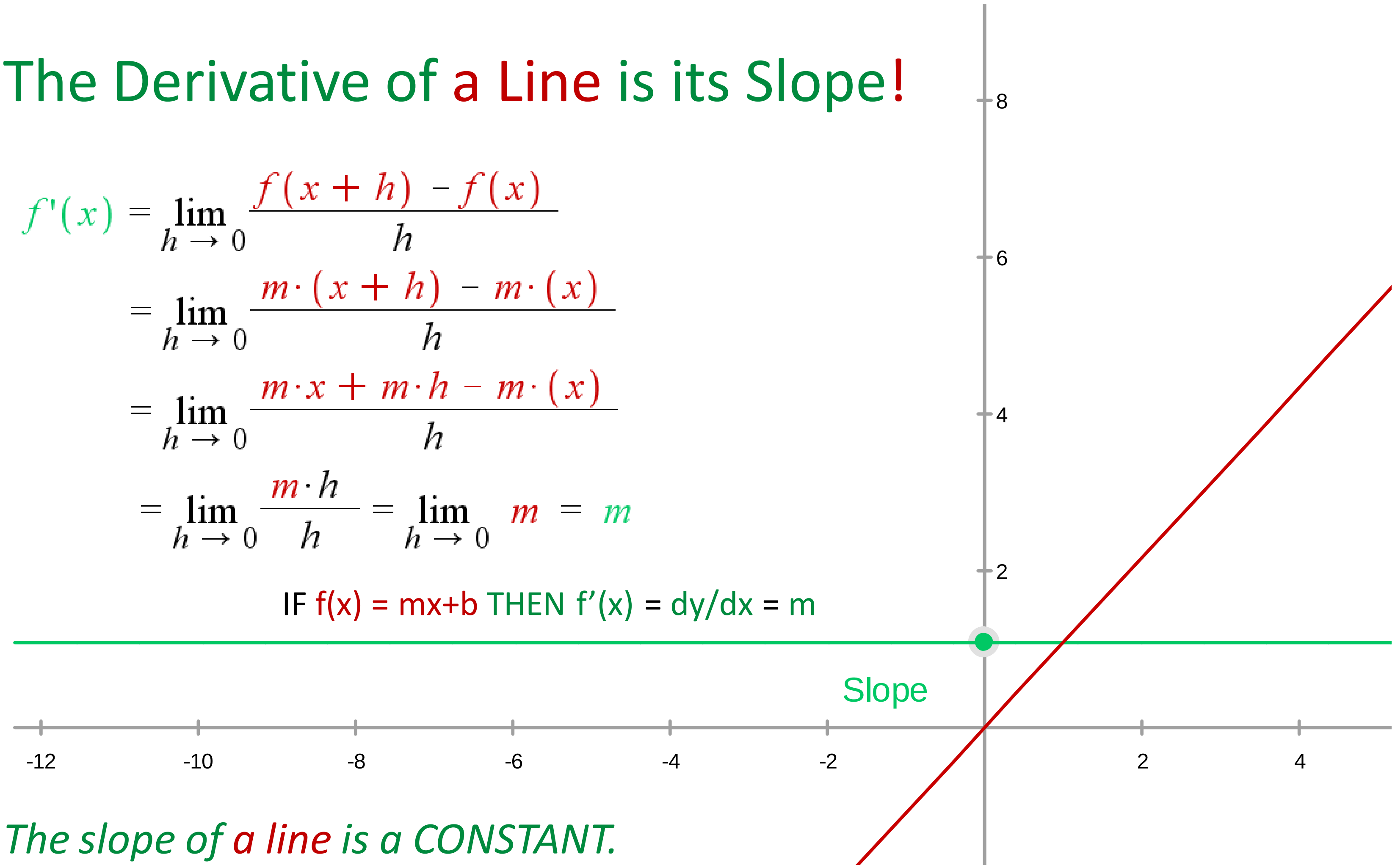
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{m\cdot(x+h) - m\cdot(x)}{h}$$

$$= \lim_{h \to 0} \frac{m\cdot x + m\cdot h - m\cdot(x)}{h}$$

$$= \lim_{h \to 0} \frac{m\cdot h}{h} = \lim_{h \to 0} m = m$$

IF f(x) = mx+b THEN f’(x) = dy/dx = m



The slope of a line is a CONSTANT.

Four Basic Operations: Lagrange Notation

$$(f + g)' = f' + g'$$
$$(f - g)' = f' - g'$$
$$(f \cdot g)' = f'g + g'f$$
$$(f / g)' = (f'g - g'f)/g^2$$

PRO – Compact

CON – Hides Differential Form (dx)

Four Basic Operations: Leibniz Notation

$$d(f + g)/dx = df/dx + dg/dx$$
$$d(f - g)/dx = df/dx - dg/dx$$
$$d(f \cdot g)/dx = df/dx \cdot g + dg/dx \cdot f$$
$$d(f / g)/dx = (df/dx \cdot g - dg/dx \cdot f)/g^2$$

PRO – Large

CON – Reveals Differential Form

Sum Rule: Two Lines

$$(f + g)' = f' + g'$$

Example:

$$f = m_1 \cdot x + b_1 \rightarrow f' = m_1$$

$$g = m_2 \cdot x + b_2 \rightarrow g' = m_2$$

$$(f + g)' = m_1 + m_2$$

Derivative of Sum is Sum of Derivatives

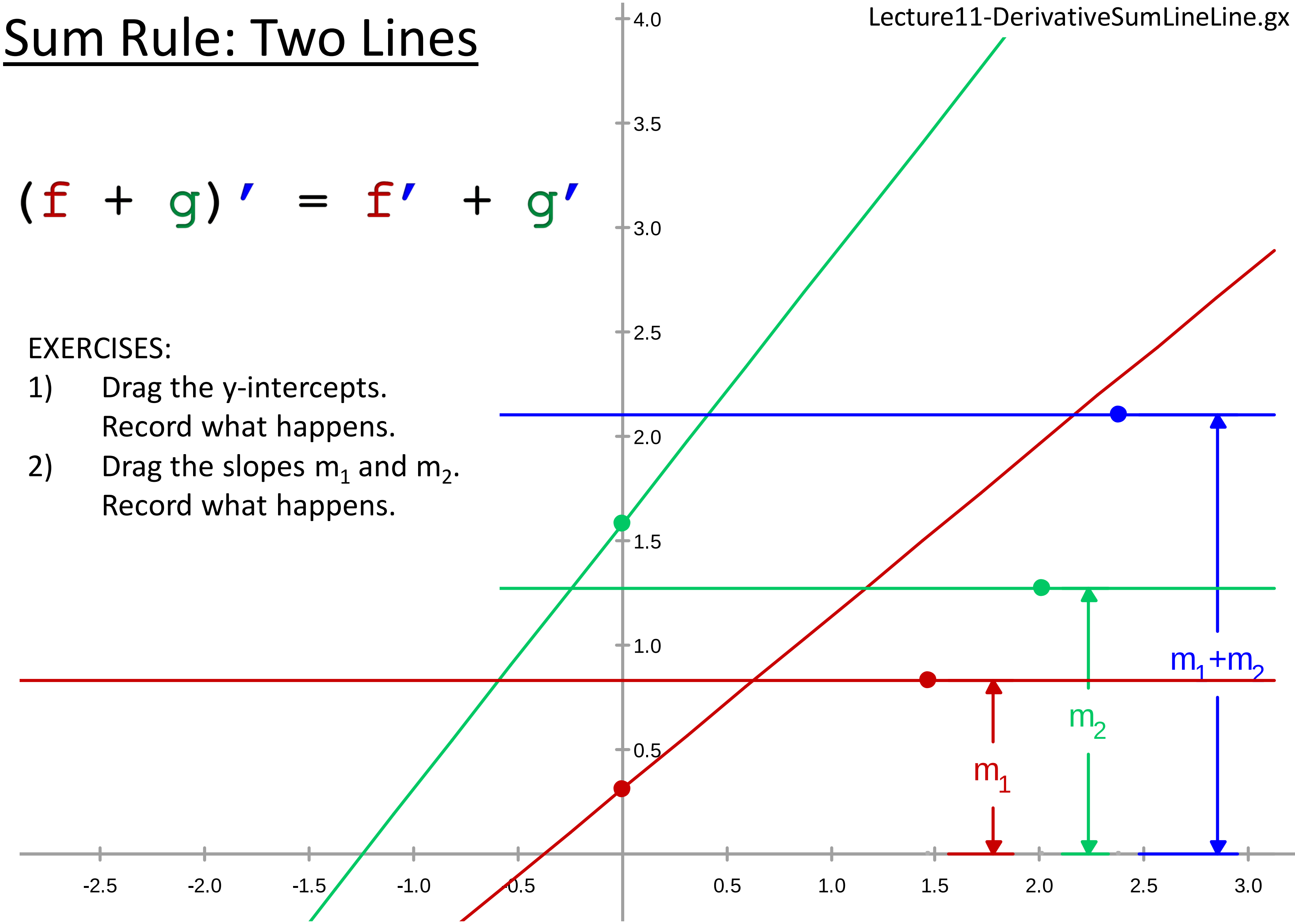
Sum Rule: Two Lines

Lecture11-DerivativeSumLineLine.gx

$$(f + g)' = f' + g'$$

EXERCISES:

1) Drag the y-intercepts. Record what happens.
2) Drag the slopes m_1 and m_2. Record what happens.



Sum Rule: Line and Parabola

$$(f + g)' = f' + g'$$

Example:

$$f = m_1 \cdot x + b_1 \rightarrow f' = m_1$$

$$g = a \cdot x^2 + m_2 \cdot x + b_2 \rightarrow g' = 2ax + m_2$$

$$(f + g)' = m_1 + 2ax + m_2$$

Derivative of Sum is Sum of Derivatives

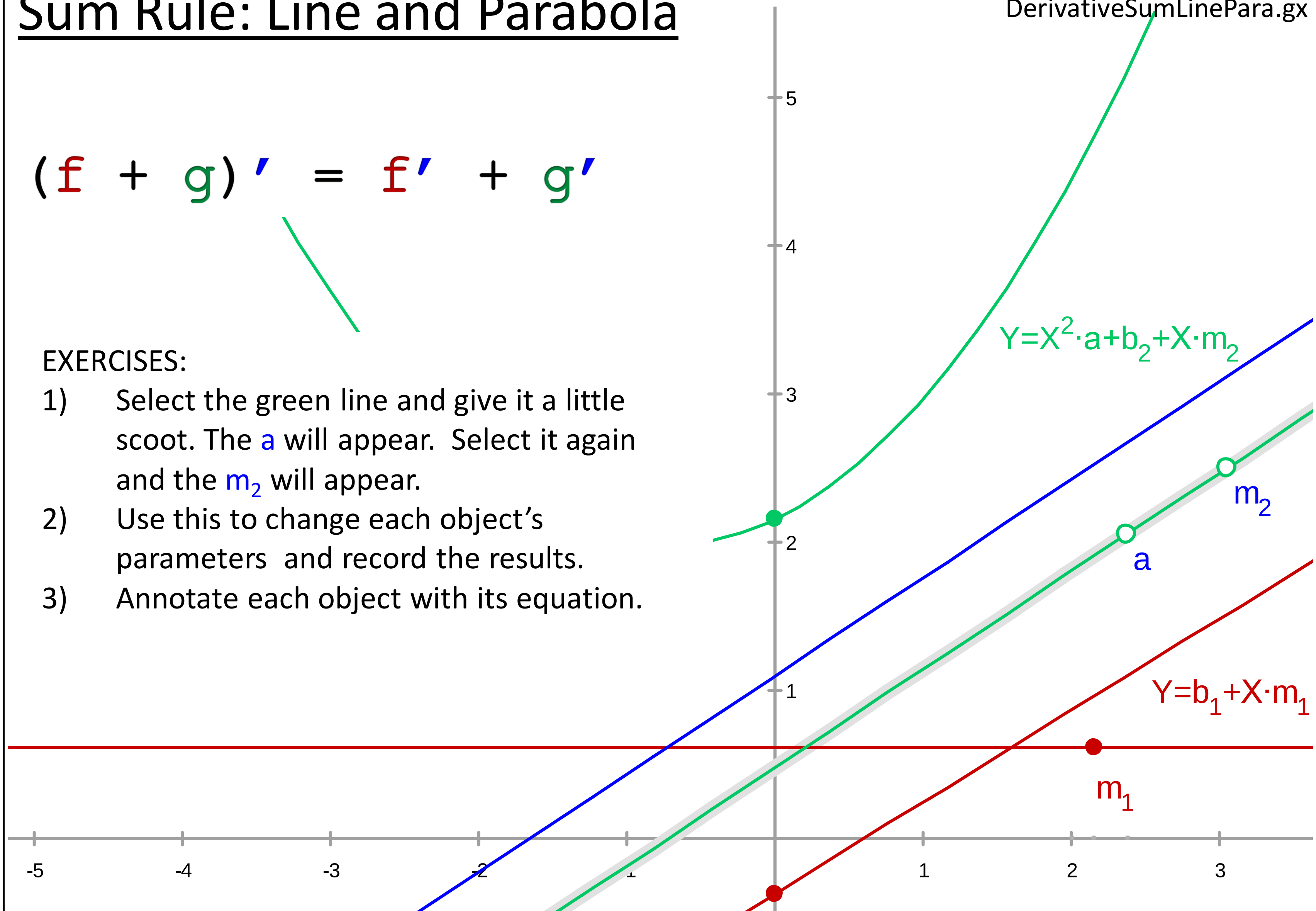
Sum Rule: Line and Parabola

Lecture11-DerivativeSumLinePara.gx

$$(f + g)' = f' + g'$$

EXERCISES:

1) Select the green line and give it a little scoot. The a will appear. Select it again and the m_2 will appear.
2) Use this to change each object's parameters and record the results.
3) Annotate each object with its equation.



Sum Rule: Two Parabolas

$$(f + g)' = f' + g'$$

Example:

$$f = a_1 \cdot x^2 + m_1 \cdot x + b_1 \rightarrow f' = 2a_1x + m_1$$

$$g = a_2 \cdot x^2 + m_2 \cdot x + b_2 \rightarrow g' = 2a_2x + m_2$$

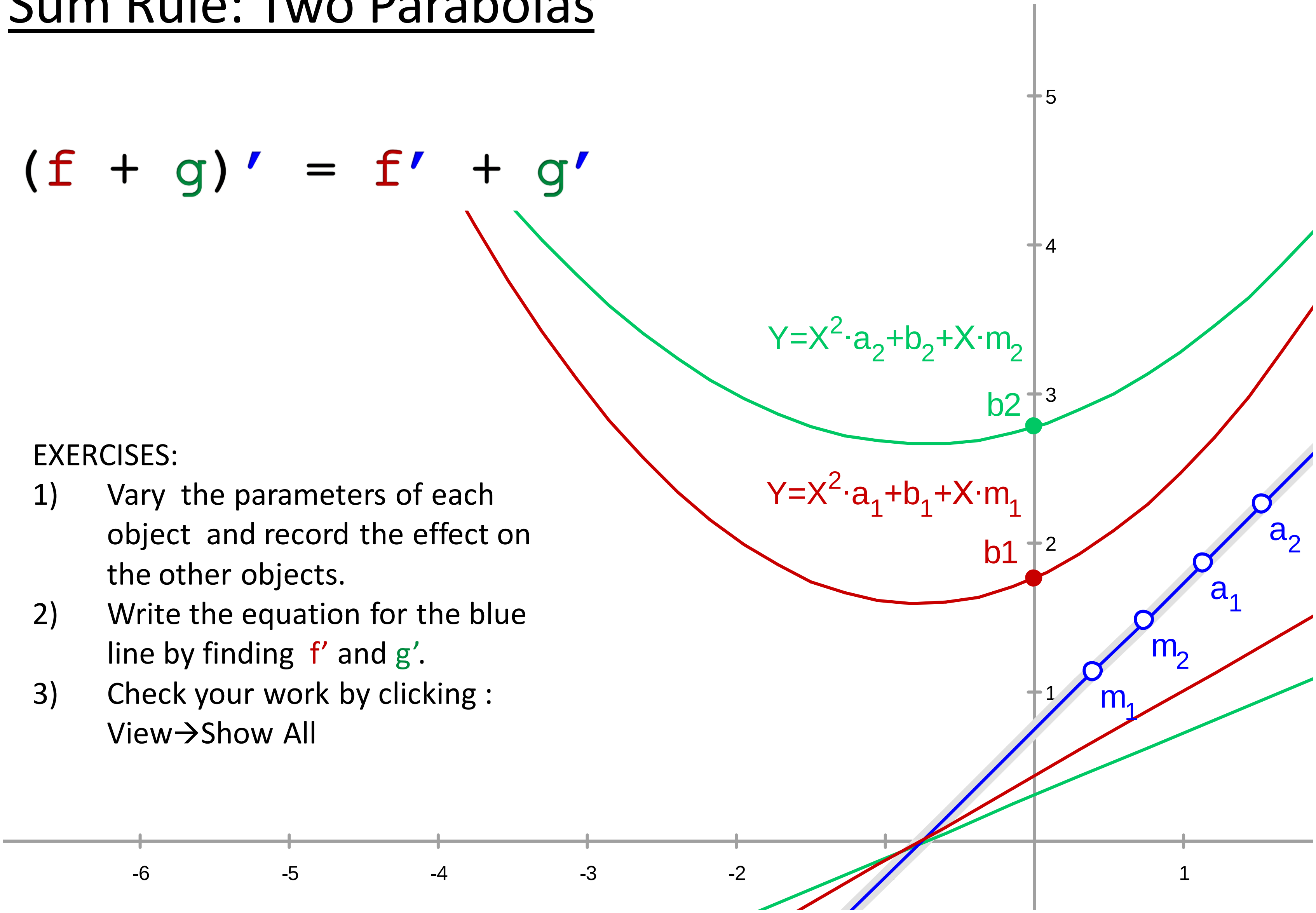
$$(f + g)' = \text{EXERCISE}$$

Derivative of Sum is Sum of Derivatives

Sum Rule: Two Parabolas

Lecture11-DerivativeSumParaPara.gx

$$(f + g)' = f' + g'$$



EXERCISES:

1) Vary the parameters of each object and record the effect on the other objects.
2) Write the equation for the blue line by finding f' and g'.
3) Check your work by clicking : View→Show All

Sum Rule: Line and Sine Wave

$(f + g)' = f' + g'$

Example:

$f = m \cdot x + b \rightarrow f' = m$

$g = a \cdot \sin(x) \rightarrow g' = a \cdot \cos(x)$

$(f + g) = m \cdot x + b + a \cdot \sin(x)$

$(f + g)' = m + 0 + a \cdot \cos(x)$

Derivative of Sum is Sum of Derivatives

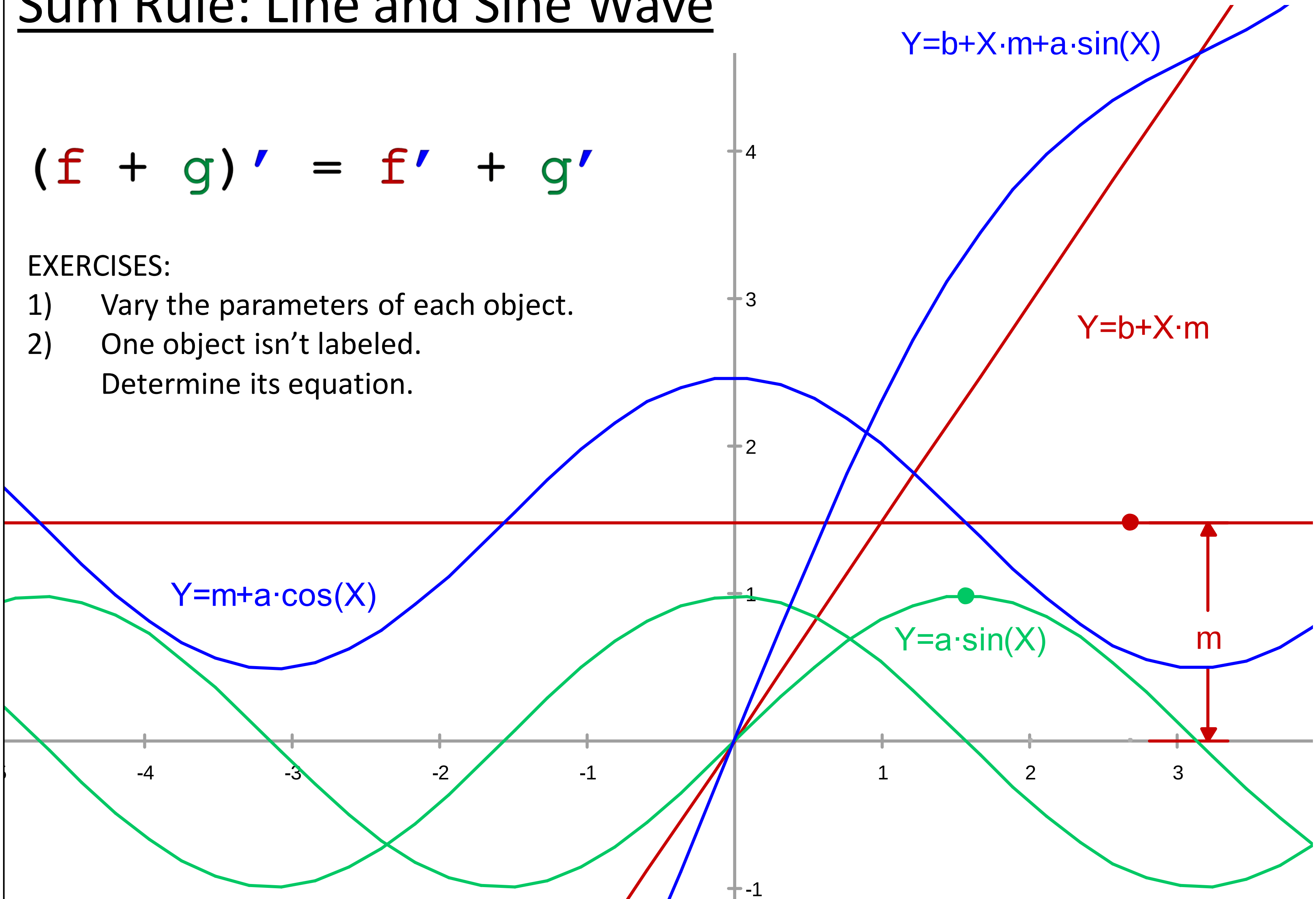
Sum Rule: Line and Sine Wave

Lecture11-DerivativeSumLineSine.gx

$$(f + g)' = f' + g'$$

EXERCISES:

1) Vary the parameters of each object.
2) One object isn't labeled. Determine its equation.



Difference Rule from Sum Rule

Prove:

$(f - g)' = f' - g'$

$(f + h)' = f' + h'$	Sum Rule
$(f - g)' = f' - g'$	Let $h = -g$
	QED

Derivative of Difference is Difference of Derivatives

Product Rule: Two Lines

$$(f \cdot g)' = f'g + g'f$$

Example:

$$f = m_1 \cdot x + b_1 \rightarrow f' = m_1$$

$$g = m_2 \cdot x + b_2 \rightarrow g' = m_2$$

$$\begin{aligned}(f \cdot g)' &= m_1(m_2 x + b_2) + m_2(m_1 x + b_1) \\ &= m_1 m_2 x + m_1 b_2 + m_2 m_1 x + m_2 b_1 \\ &= 2 \cdot m_1 m_2 x + m_1 b_2 + m_2 b_1\end{aligned}$$

First Prime x *Second* + *Second Prime* x *First*

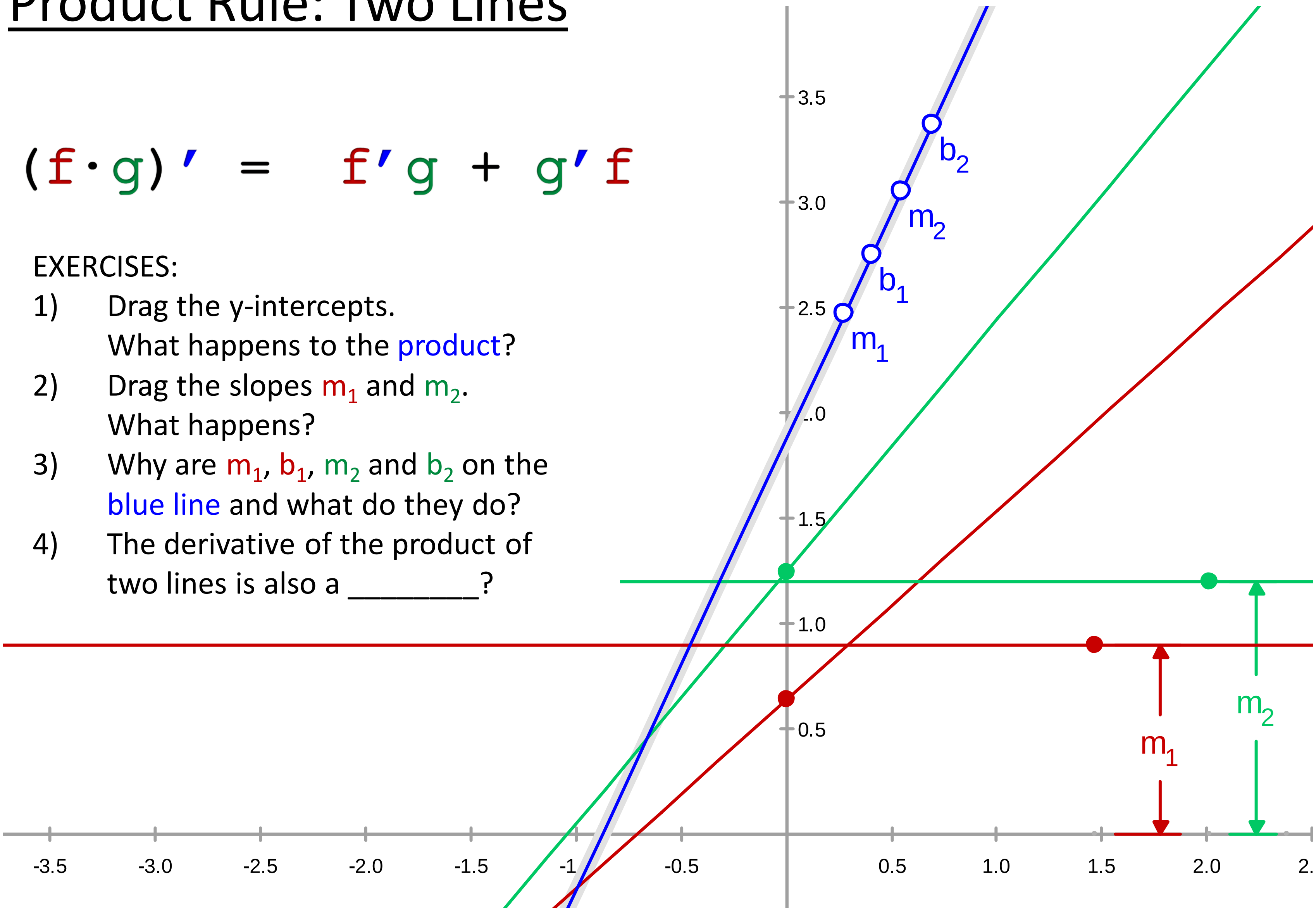
Product Rule: Two Lines

Lecture11-DerivativeProdLineLine.gx

$$(f \cdot g)' = f'g + g'f$$

EXERCISES:

1) Drag the y-intercepts.
   What happens to the product?
2) Drag the slopes m_1 and m_2.
   What happens?
3) Why are m_1, b_1, m_2 and b_2 on the blue line and what do they do?
4) The derivative of the product of two lines is also a ________?



Product Rule: Two Parabolas

$$(f \cdot g)' = f'g + g'f$$

Example:

$$f = a_1 \cdot x^2 + m_1 \cdot x + b_1 \rightarrow f' = 2a_1 x + m_1$$

$$g = a_2 \cdot x^2 + m_2 \cdot x + b_2 \rightarrow g' = 2a_2 x + m_2$$

$$(f \cdot g)' = (2a_1 x + m_1) \cdot (a_2 x^2 + m_2 x + b_2) + (2a_2 x + m_2) \cdot (a_1 x^2 + m_1 x + b_1)$$

$$= 4a_1 a_2 x^3 + 3(a_1 m_2 + a_2 m_1) x^2 + 2(m_1 m_2 + a_1 b_2 + a_2 b_1) x + b_1 m_2 + b_2 m_1$$

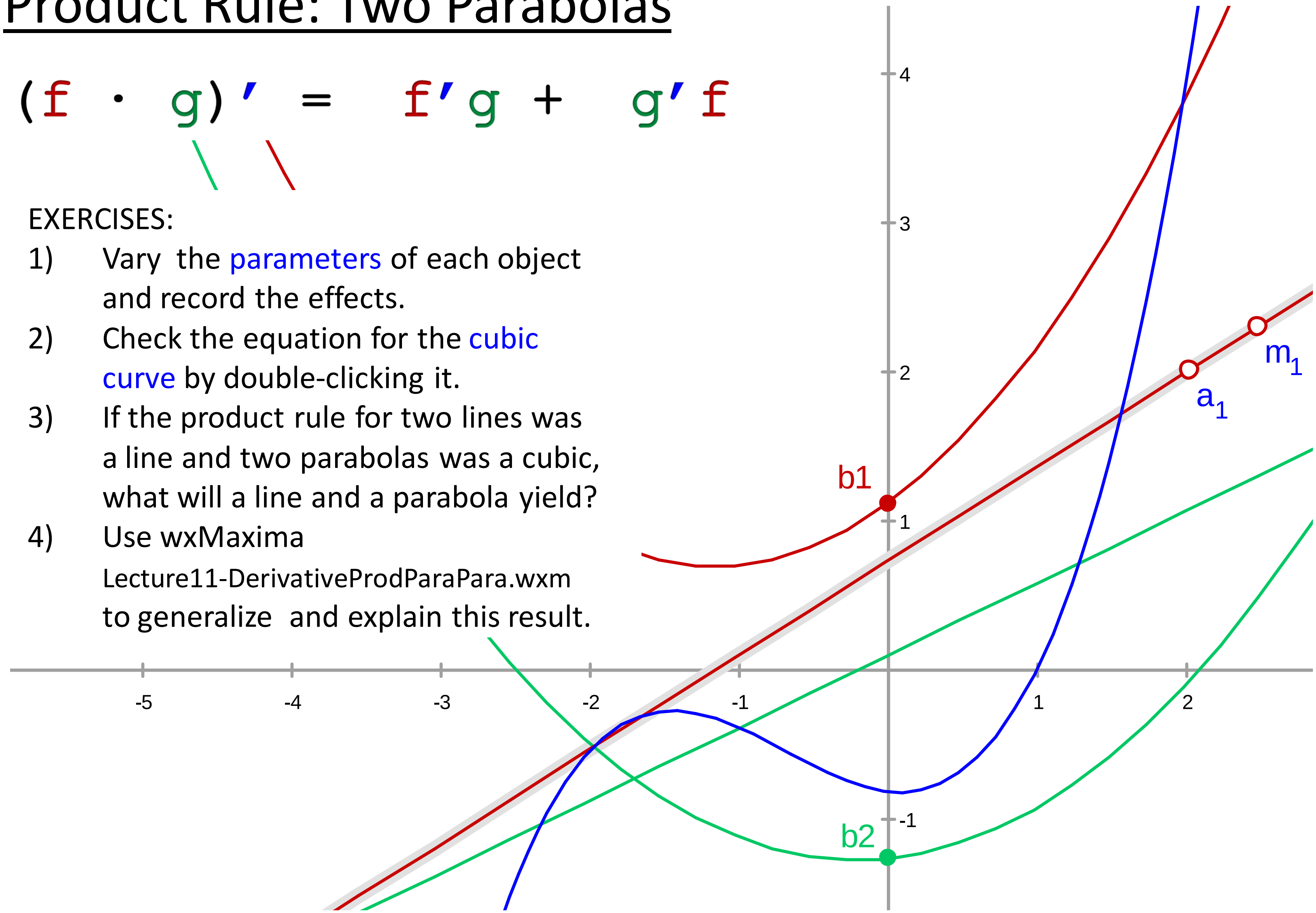
Product Rule: Two Parabolas

Lecture11-DerivativeProdParaPara.gx

$$(f \cdot g)' = f'g + g'f$$

EXERCISES:

1) Vary the parameters of each object and record the effects.
2) Check the equation for the cubic curve by double-clicking it.
3) If the product rule for two lines was a line and two parabolas was a cubic, what will a line and a parabola yield?
4) Use wxMaxima Lecture11-DerivativeProdParaPara.wxm to generalize and explain this result.



Product Rule: Line and Sine Wave

$$(f \cdot g)' = f'g + g'f$$

Example:

$$f = m \cdot x + b \rightarrow f' = m$$
$$g = \sin(x) \rightarrow g' = \cos(x)$$

$$(f \cdot g)' = f'g + g'f$$
$$= m \sin(x) + \cos(x)(m \cdot x + b)$$

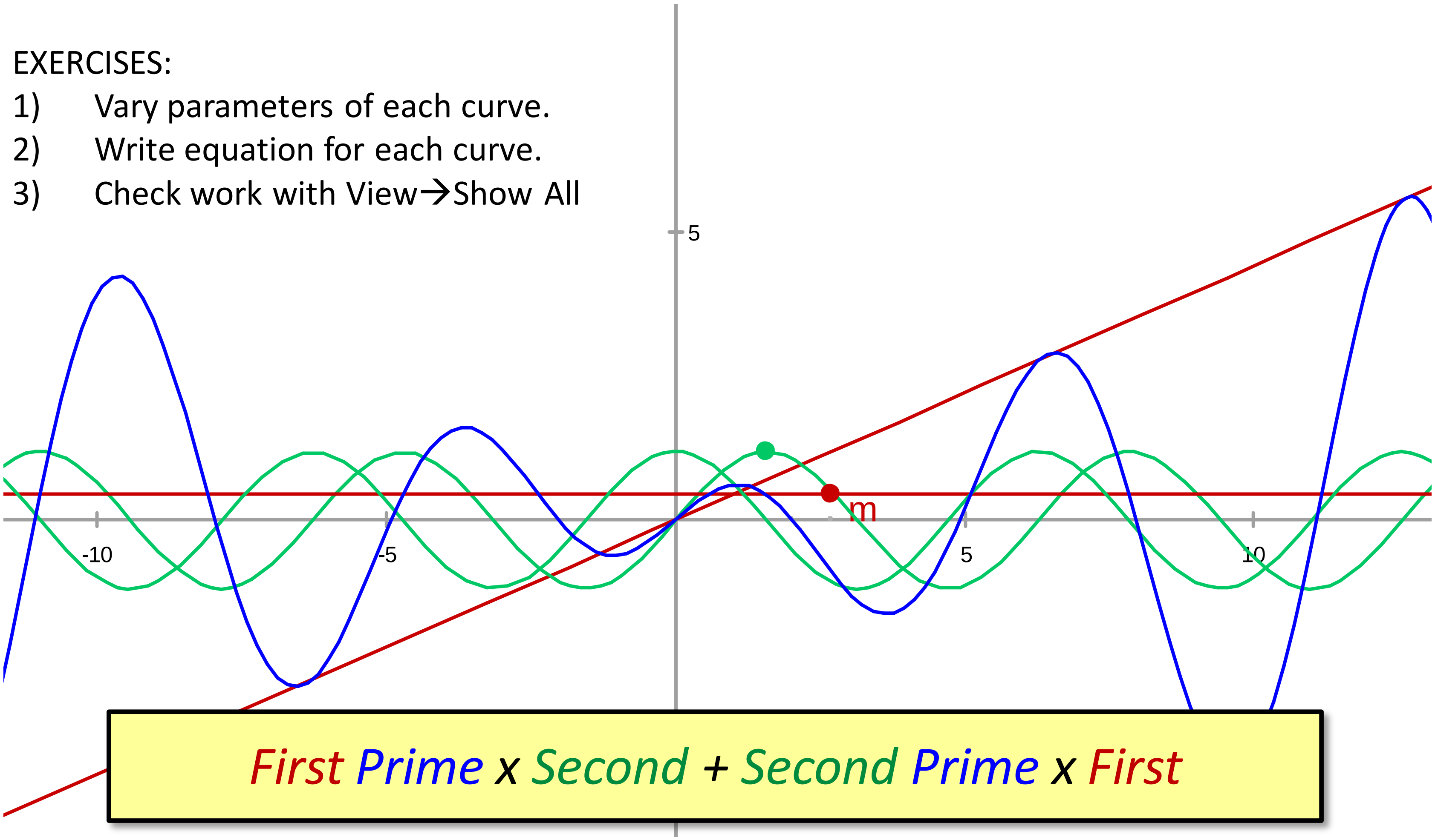
First Prime x *Second* + *Second Prime* x *First*

Product Rule: Line and Sine

Lecture11-DerivativeProdLineSine.gx

$$(f \cdot g)' = f'g + g'f$$

EXERCISES:
1) Vary parameters of each curve.
2) Write equation for each curve.
3) Check work with View→Show All



First Prime x Second + Second Prime x First

Product Rule: Two "Sine" Waves

$$(f \cdot g)' = f'g + g'f$$

Example:

$$f = \sin(x) - a \rightarrow f' = \cos(x)$$

$$g = \cos(x) - b \rightarrow g' = -\sin(x)$$

$$\begin{aligned}(f \cdot g)' &= f'g + g'f \\ &= \cos(x)(\cos(x) - b) + \\ &\quad -\sin(x)(\sin(x) - a) \\ &= -\sin(x)^2 + a \cdot \sin(x) + \\ &\quad \cos(x)^2 - b \cdot \cos(x)\end{aligned}$$

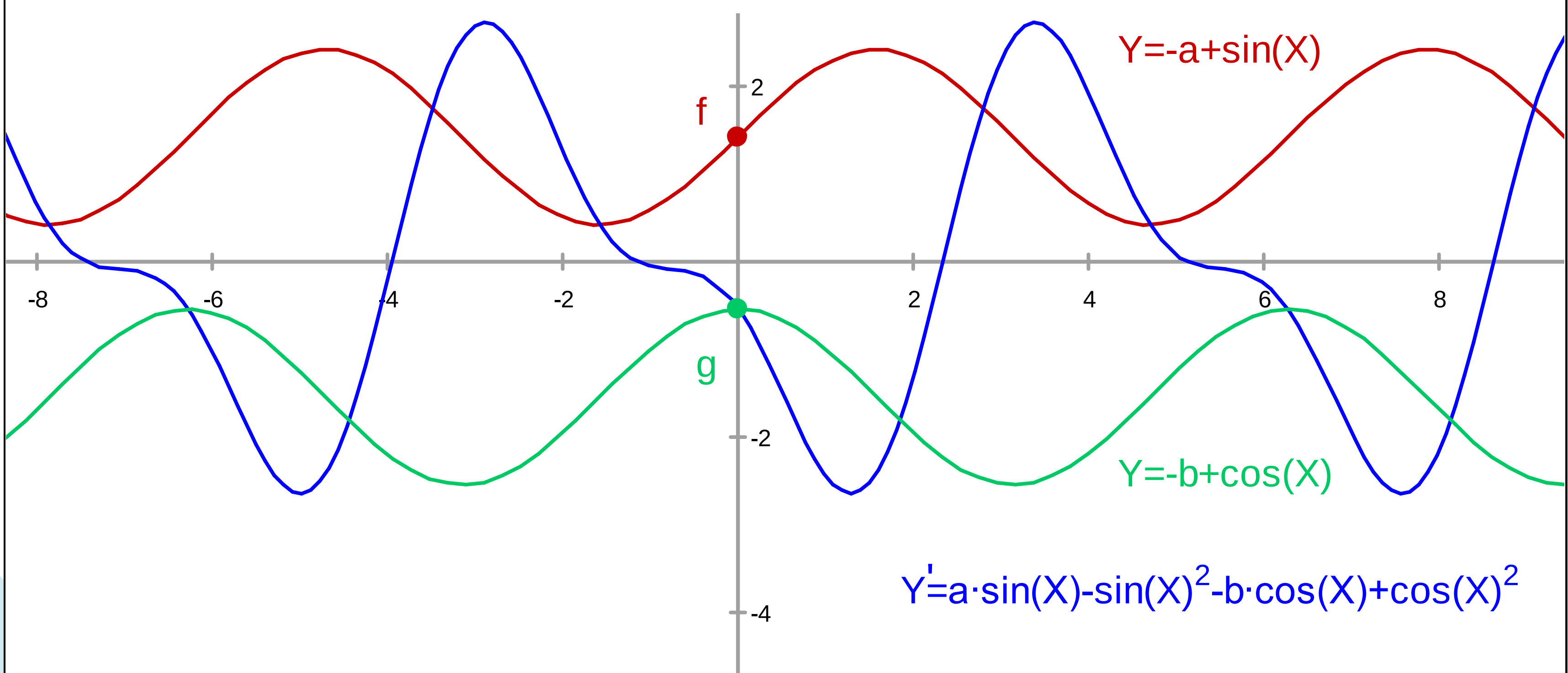
Product Rule: Two Sine Waves

Lecture11-DerivativeProdSinCos.gx

$$(f \cdot g)' = f'g + g'f$$

EXERCISES:

1) Vary parameters of each curve.
2) Why are g and $(f \cdot g)'$ coincident at x = 0?



Quotient Rule: Two Lines

$(f/g)' = (f'g - g'f)/g^2$

Example:

$f = m_1 \cdot x + b_1 \rightarrow f' = m_1$

$g = m_2 \cdot x + b_2 \rightarrow g' = m_2$

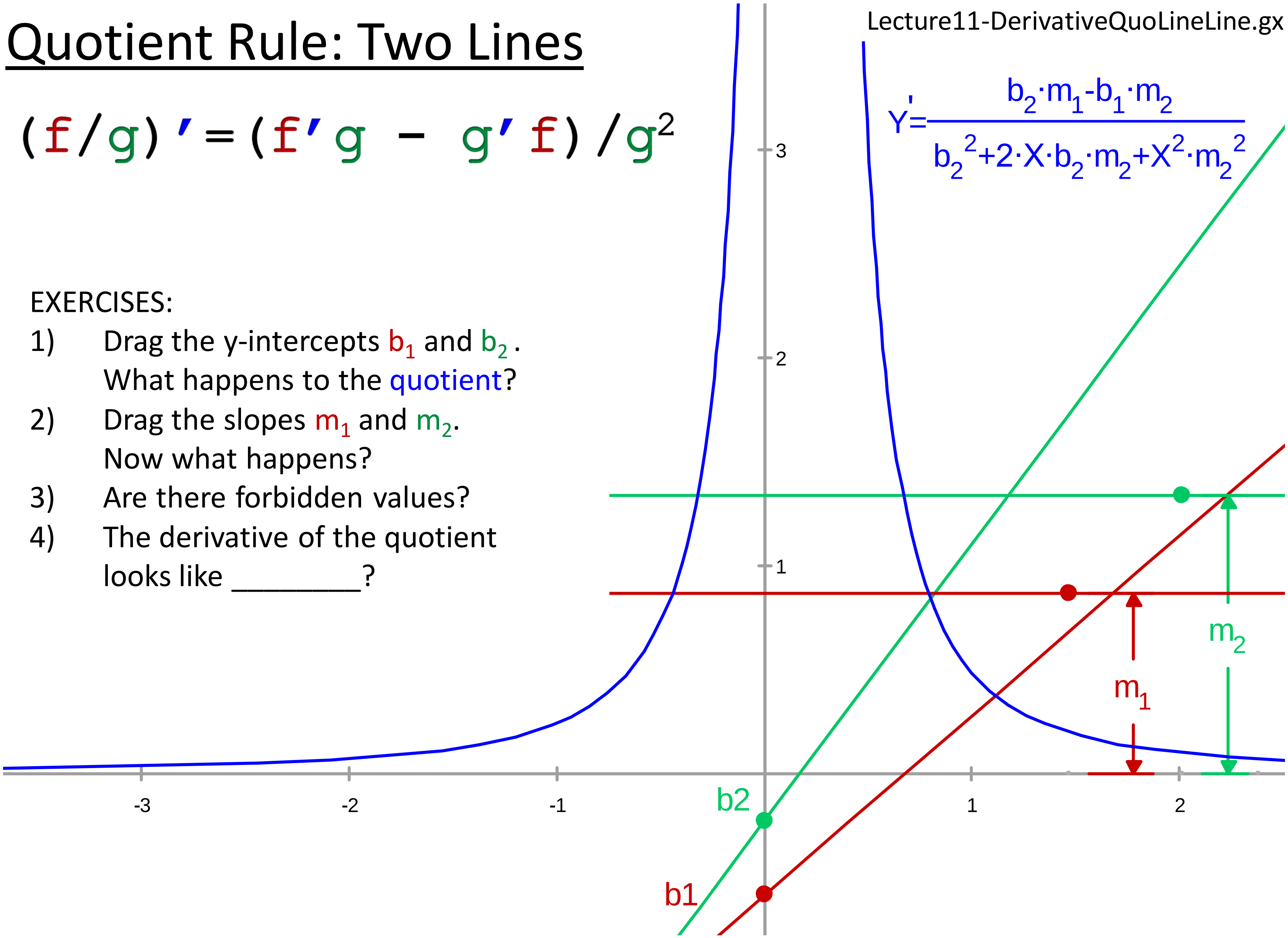
$$
\begin{aligned}
(f/g)' &= (f'g - g'f)/g^2 \\
&= (m_1(m_2x+b_2)-m_2(m_1x+b_1))/(m_2x+b_2)^2 \\
&= (b_2m_1-b_1m_2)/(m_2^2x^2+2b_2m_2x+b_2^2)
\end{aligned}
$$

Quotient Rule: Two Lines

$(f/g)' = (f'g - g'f)/g^2$

EXERCISES:

1) Drag the y-intercepts b_1 and b_2. What happens to the quotient?
2) Drag the slopes m_1 and m_2. Now what happens?
3) Are there forbidden values?
4) The derivative of the quotient looks like ________?



Quotient Rule: Two Parabolas

Lecture11-DerivativeQuoParaPara.wxm

$(f/g)' = (f'g - g'f)/g^2$

Example:

$f = a_1 \cdot x^2 + b_1 \cdot x + c_1 \rightarrow f' = 2a_1x+b_1$

$g = a_2 \cdot x^2 + b_2 \cdot x + c_2 \rightarrow g' = 2a_2x+b_2$

$$(f/g)' = \frac{\left((2a_1x+b_1)(a_2x^2+b_2x+c_2) - (2a_2x+b_2)(a_1x^2+b_1x+c_1) \right)}{(a_2 \cdot x^2 + b_2 \cdot x + c_2)^2}$$

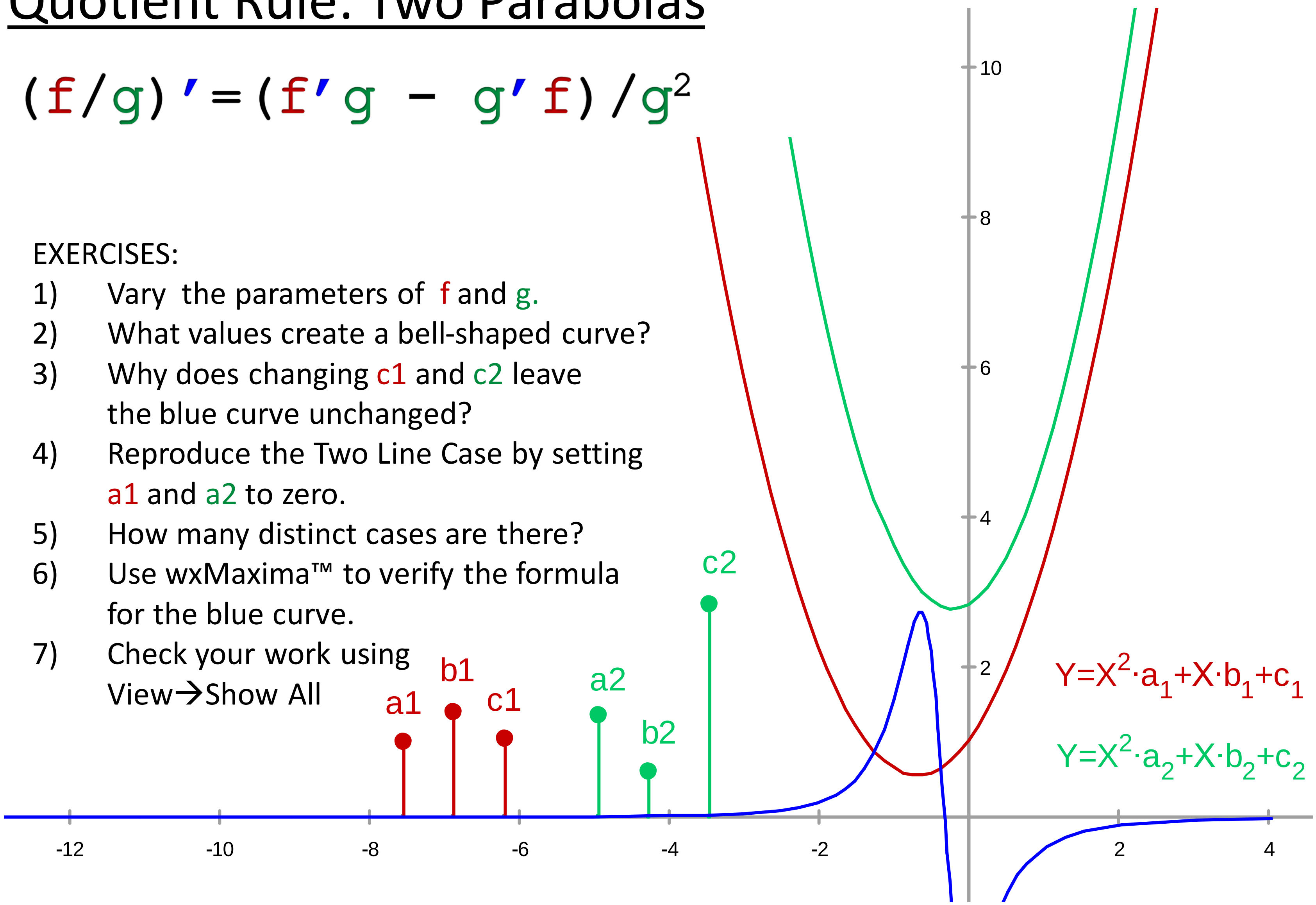
Lecture11-DerivativeQuoParaPara.gx

Quotient Rule: Two Parabolas

$$(f/g)' = (f'g - g'f)/g^2$$

EXERCISES:

1) Vary the parameters of f and g.
2) What values create a bell-shaped curve?
3) Why does changing c1 and c2 leave the blue curve unchanged?
4) Reproduce the Two Line Case by setting a1 and a2 to zero.
5) How many distinct cases are there?
6) Use wxMaxima™ to verify the formula for the blue curve.
7) Check your work using View→Show All



Quotient Rule: Sine and Line = Sinc Function

$(f/g)' = (f'g - g'f)/g^2$

Example:

$f = a \cdot \sin(x) \rightarrow f' = a \cdot \cos(x)$

$g = x \rightarrow g' = 1$

$(f/g) = a \cdot \sin(x)/x = a \cdot \mathrm{sinc}(x)$

$(f/g)' = a(f'g - g'f)/g^2$

$= a(\cos(x) \cdot x - 1 \cdot \sin(x))/(x)^2$

$= a \cos(x)/x - a \sin(x)/x^2$

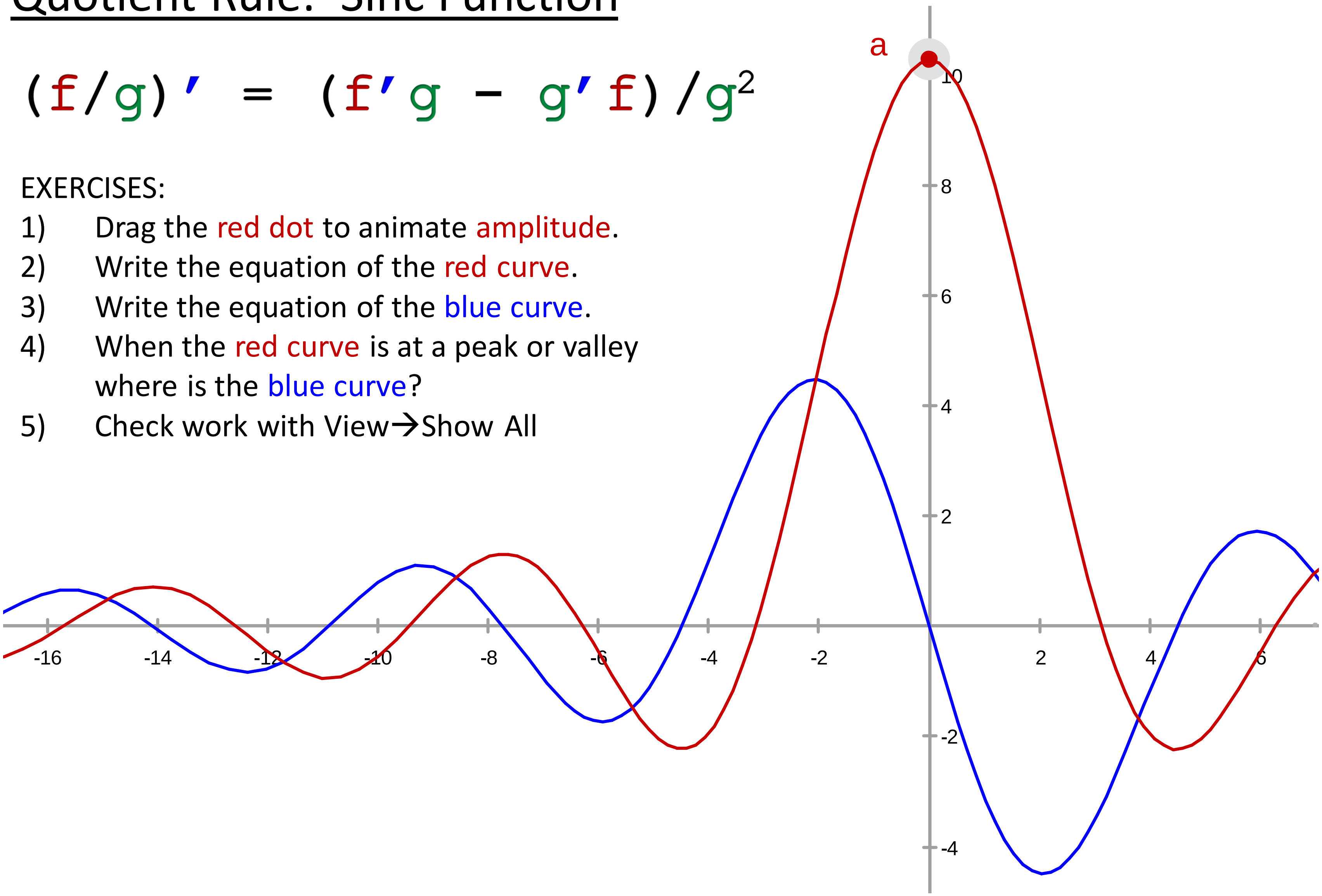
Quotient Rule: Sinc Function

Lecture11-DerivativeQuoSineLine.gx

$$(f/g)' = (f'g - g'f)/g^2$$

EXERCISES:

1) Drag the red dot to animate amplitude.
2) Write the equation of the red curve.
3) Write the equation of the blue curve.
4) When the red curve is at a peak or valley where is the blue curve?
5) Check work with View→Show All



The Chain Rule:

Given f(g(x)) then f′(x) =

df/dx = df/dg·dg/dx

iff f and g are differentiable.

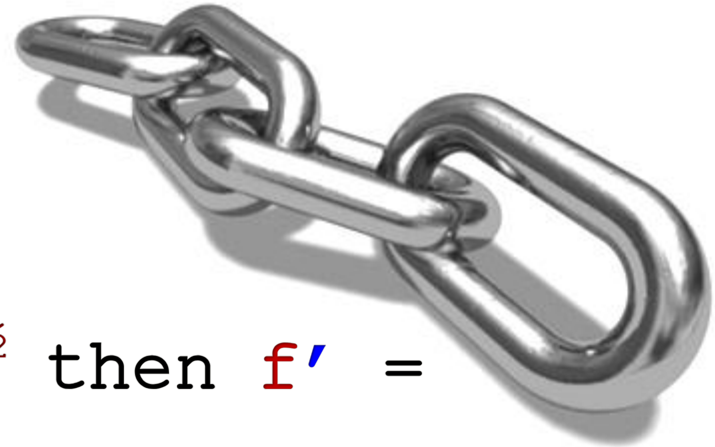
The Chain Rule is recursive!

The Chain Rule:

Given f(g(x)) then f′(x) =

$$df/dx = \underbrace{df/dg}_{\text{Outer Expression}} \cdot \underbrace{dg/dx}_{\text{Inner Expression}}$$

The Chain Rule:



Example 1: $f = (r^2-x^2)^{1/2}$ then $f' =$

$$df/dx = df/dg \cdot dg/dx$$

Let $g = (r^2-x^2)$

then $dg/dx = -2x$

and $df/dg = \frac{1}{2}\,(g)^{-1/2}$

so $df/dx = \frac{1}{2}\,(r^2-x^2)^{-1/2} \cdot (-2x)$

$= -x(r^2-x^2)^{-1/2}$

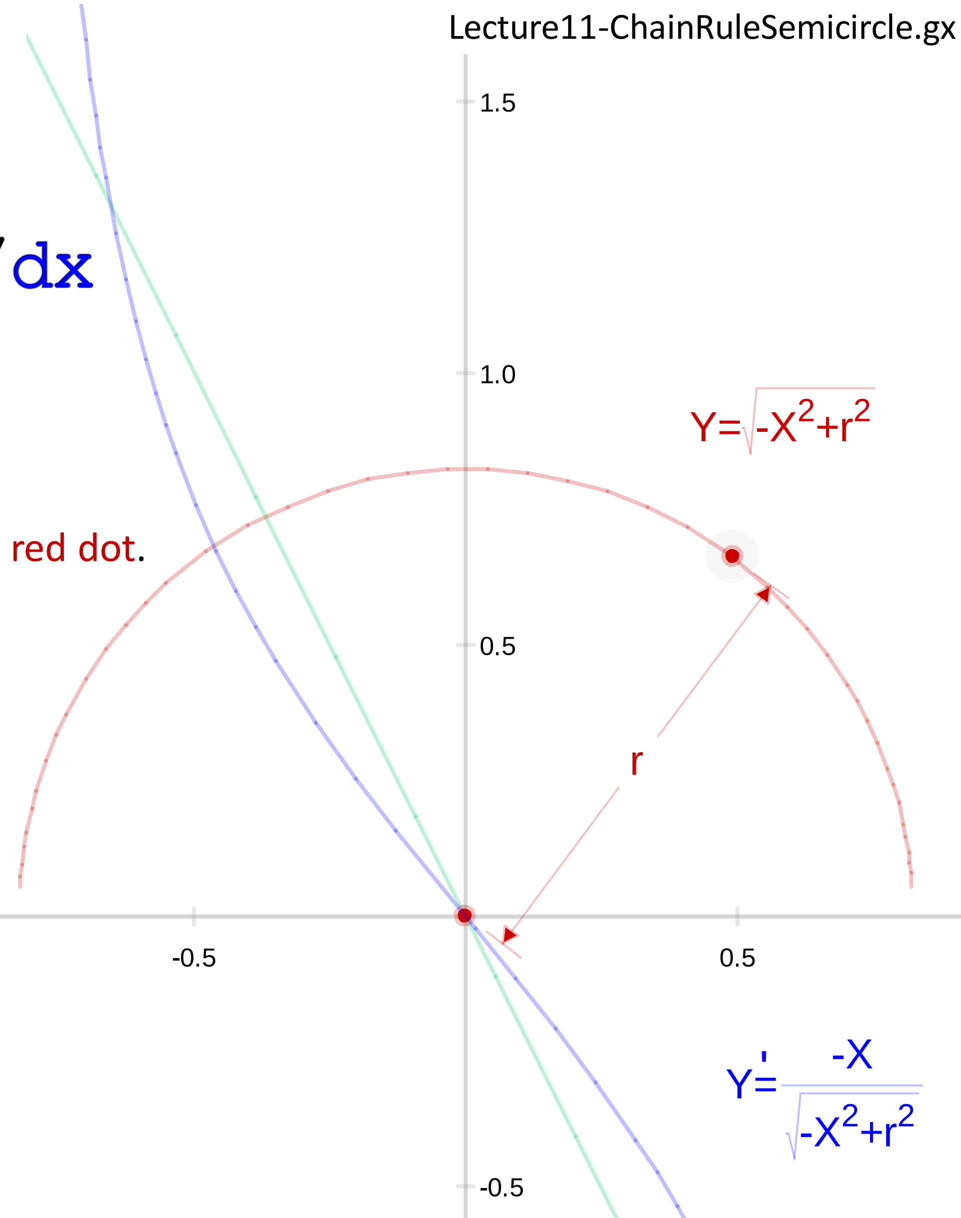
The Chain Rule

Lecture11-ChainRuleSemicircle.gx

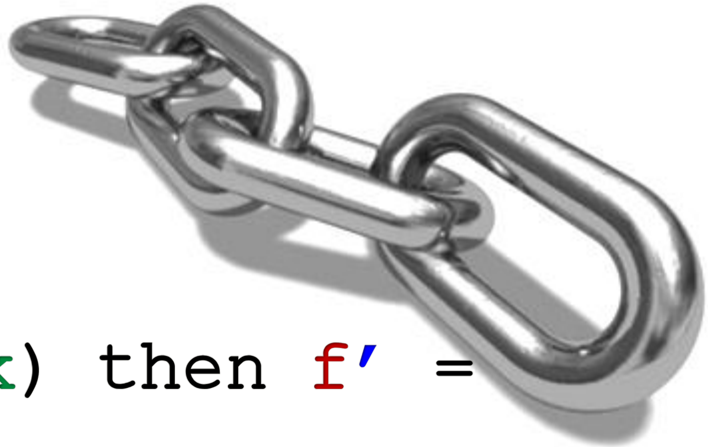
$$df/dx = df/dg \cdot dg/dx$$

EXERCISES:

1) Which object represents dg/dx?
2) Which object represents df/dx?
3) Change the radius by dragging the red dot.
4) What happens to the derivative?
5) What happens to dg/dx?
6) Explain a reason for your findings.



The Chain Rule:



Example 2: $f = \sin(x^3/k)$ then $f' =$

$$df/dx = df/dg \cdot dg/dx$$

Let $g = (x^3/k)$

then $dg/dx = 3x^2/k$

and $df/dg = \cos(g)$

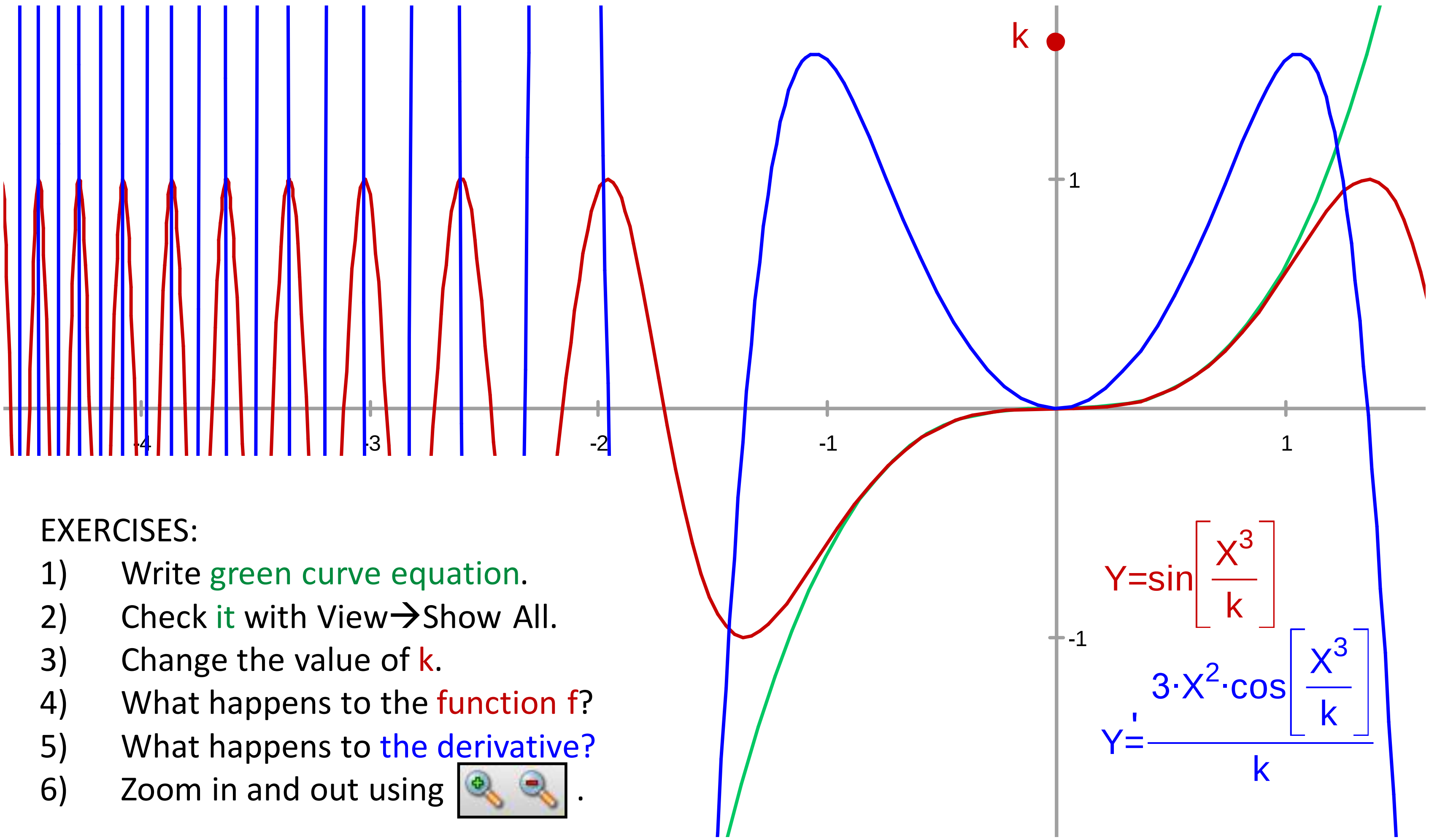
so $df/dx = \cos(x^3/k) \cdot (3x^2/k)$

$= (3x^2/k) \cdot \cos(x^3/k)$

The Chain Rule

Lecture11-ChainRuleSinXCubed.gx

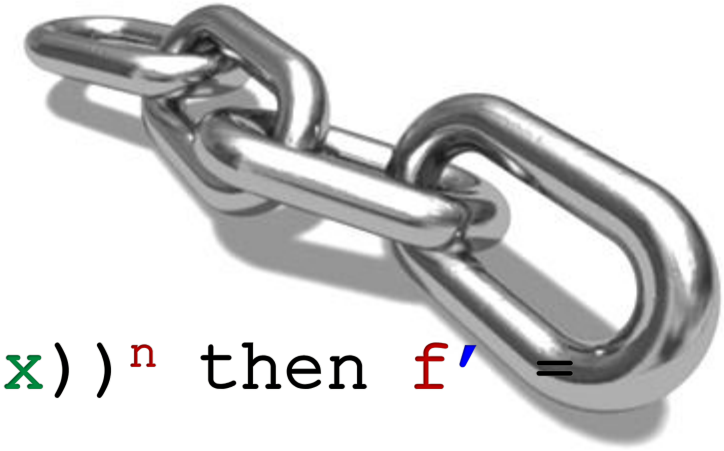
$$df/dx = df/dg \cdot dg/dx$$



EXERCISES:

1) Write green curve equation.
2) Check it with View→Show All.
3) Change the value of k.
4) What happens to the function f?
5) What happens to the derivative?
6) Zoom in and out using .

The Chain Rule:



Example 3: $f = (x+\sin(x))^n$ then $f' =$

$$df/dx = df/dg \cdot dg/dx$$

Let $g = (x+\sin(x))$

then $dg/dx = (1+\cos(x))$

and $df/dg = n(g(x))^{n-1}$

so $df/dx = (n(g(x))^{n-1}) \cdot (1+\cos(x))$

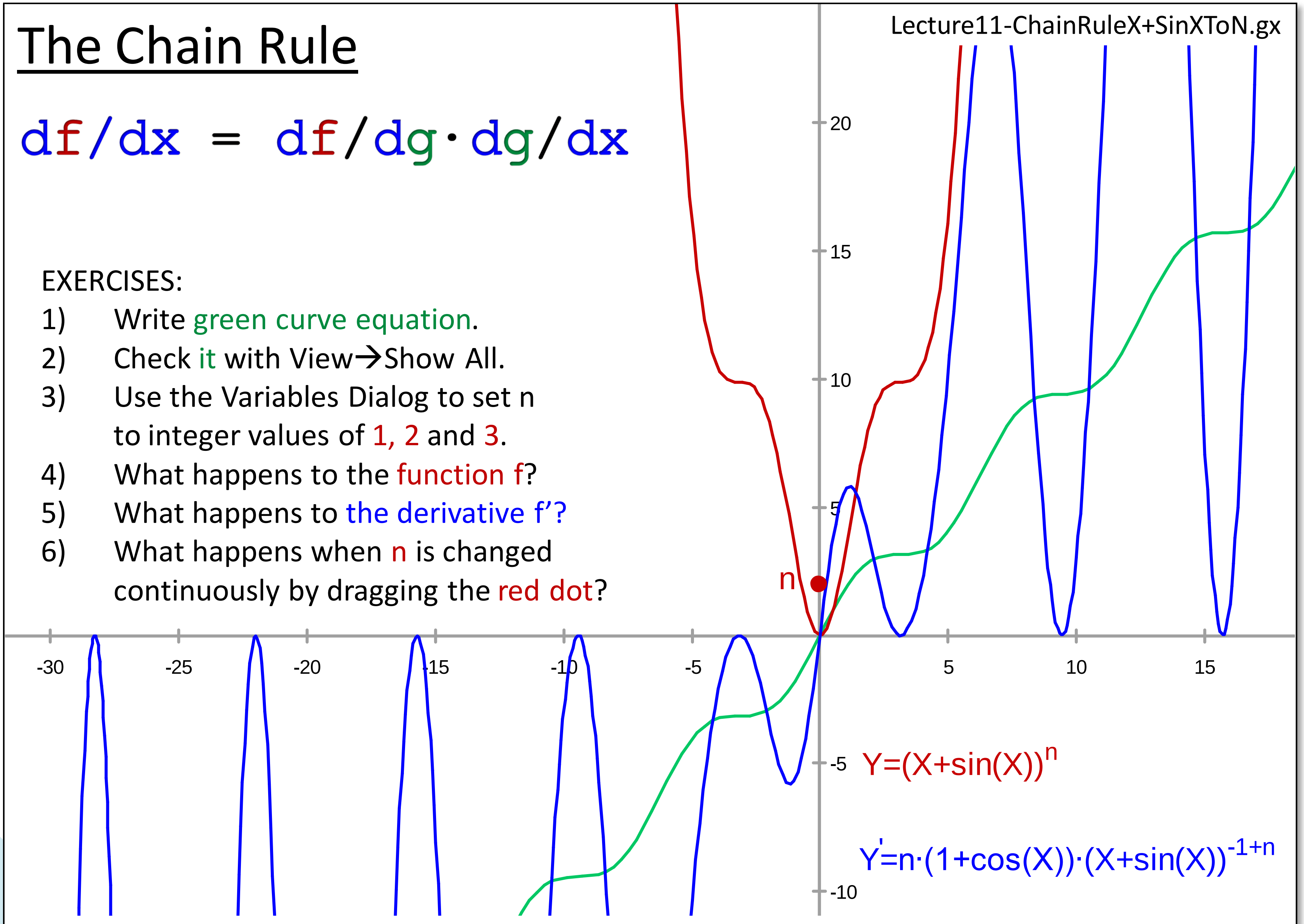
$= n(1+\cos(x)) \cdot (x+\sin(x))^{n-1}$

The Chain Rule

$$df/dx = df/dg \cdot dg/dx$$

EXERCISES:

1) Write green curve equation.
2) Check it with View→Show All.
3) Use the Variables Dialog to set n to integer values of 1, 2 and 3.
4) What happens to the function f?
5) What happens to the derivative f'?
6) What happens when n is changed continuously by dragging the red dot?



The Chain Rule: Recursion

Example 4:

$$f = A\cdot\sin(\cos(\sin(x)))$$

$$f' = df/dx = df/dg\cdot dg/dh\cdot dh/dx$$

Let

$$h = \sin(x) \rightarrow dh/dx = \cos(x)$$

$$g = \cos(h) \rightarrow dg/dh = -\sin(h)$$

$$f = A\cdot\sin(g) \rightarrow df/dg = A\cdot\cos(g)$$

Thus

$$df/dx = (A\cdot\cos(g))\cdot(-\sin(h))\cdot(\cos(x))$$

$$= -A\cdot\cos(x)\sin(\sin(x))\cos(\cos(\sin(x)))$$

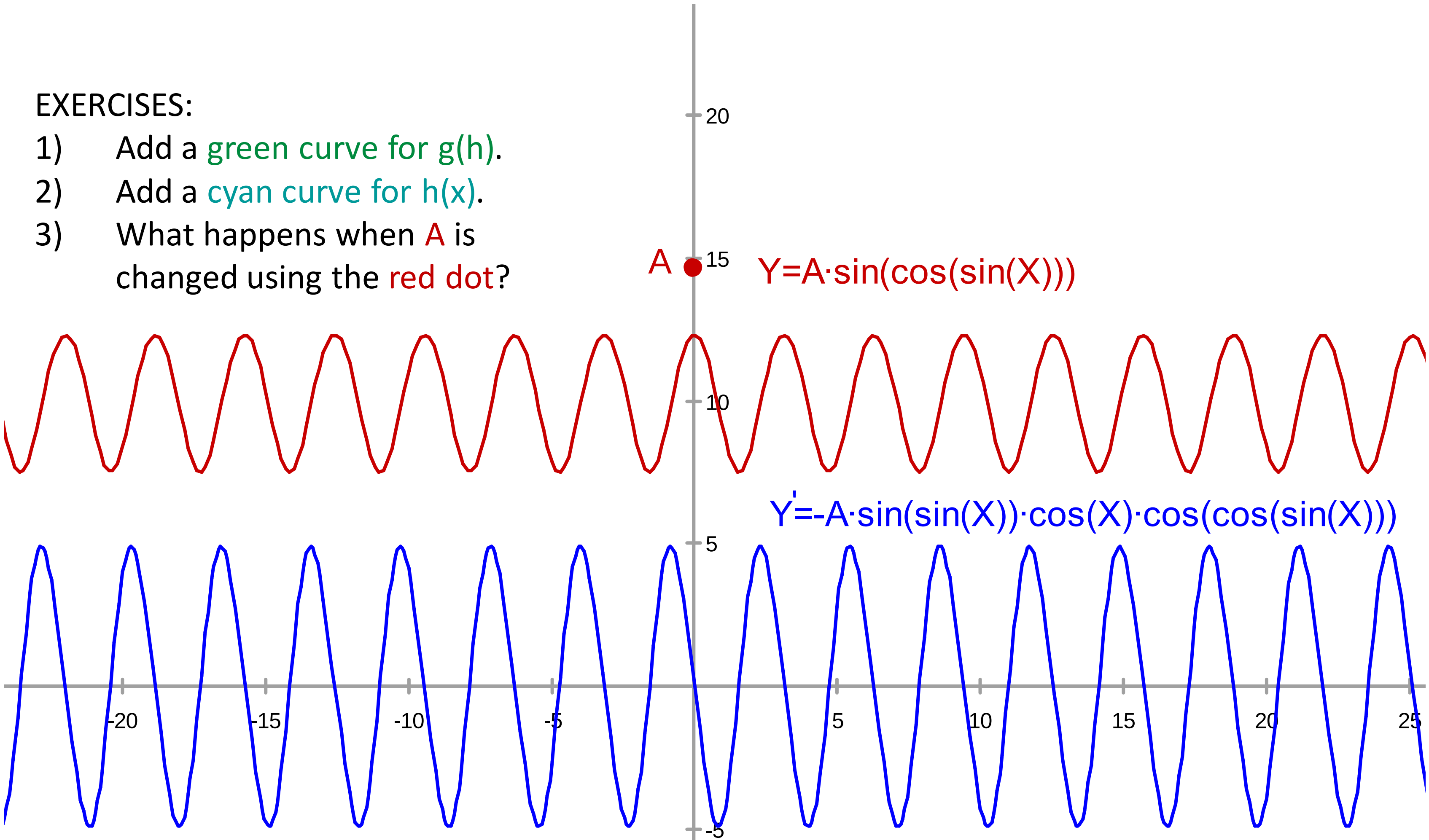
The Chain Rule

Lecture11-ChainRuleRecursive.gx

df/dx = df/dg·dg/dh·dh/dx

EXERCISES:

1) Add a green curve for g(h).
2) Add a cyan curve for h(x).
3) What happens when A is changed using the red dot?



Chain Rule with Power Rule

Example 5: $f = u^n$

$u = \sin(x)$

$$f' = df/dx = df/du \cdot du/dx$$

$$df/dx = d(u^n)/dx = n \cdot u^{n-1} \cdot du/dx$$

$$= n \cdot u^{n-1} \cdot \cos(x)$$

$$= n \cdot \sin(x)^{n-1} \cdot \cos(x)$$

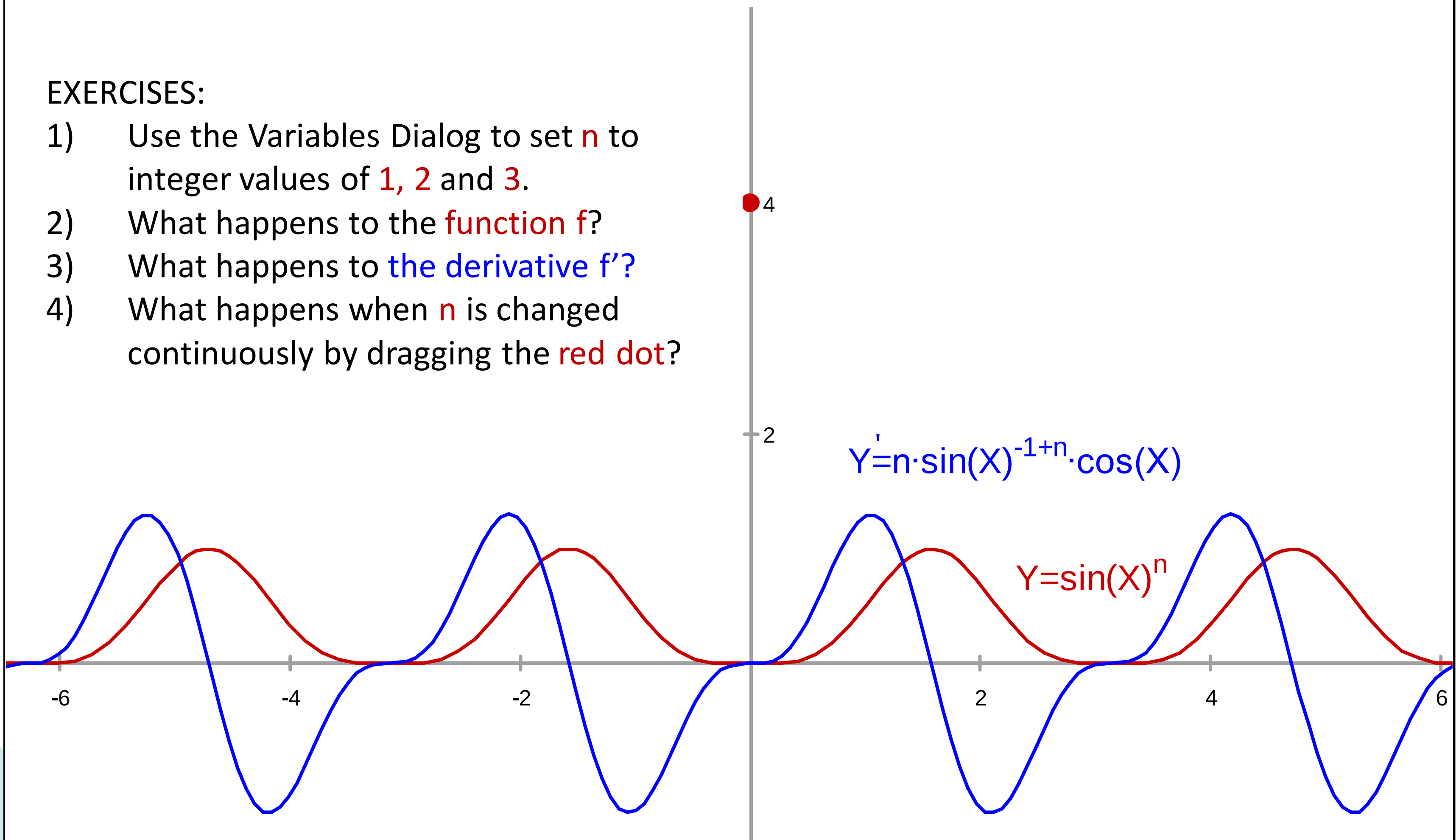
Chain Rule with Power Rule

Lecture11-ChainRulePowerRule.gx

$$df/dx = df/du \cdot du/dx$$

EXERCISES:

1) Use the Variables Dialog to set n to integer values of 1, 2 and 3.
2) What happens to the function f?
3) What happens to the derivative f'?
4) What happens when n is changed continuously by dragging the red dot?



Definition of Differentiability

f(x) is differentiable at x=a
if f′(a) exists.

f(x) is differentiable on interval (a,b) if
it is differentiable on every x in (a,b).

Theorem

if f(x) is differentiable at x = a,
f(x) is continuous at x = a.

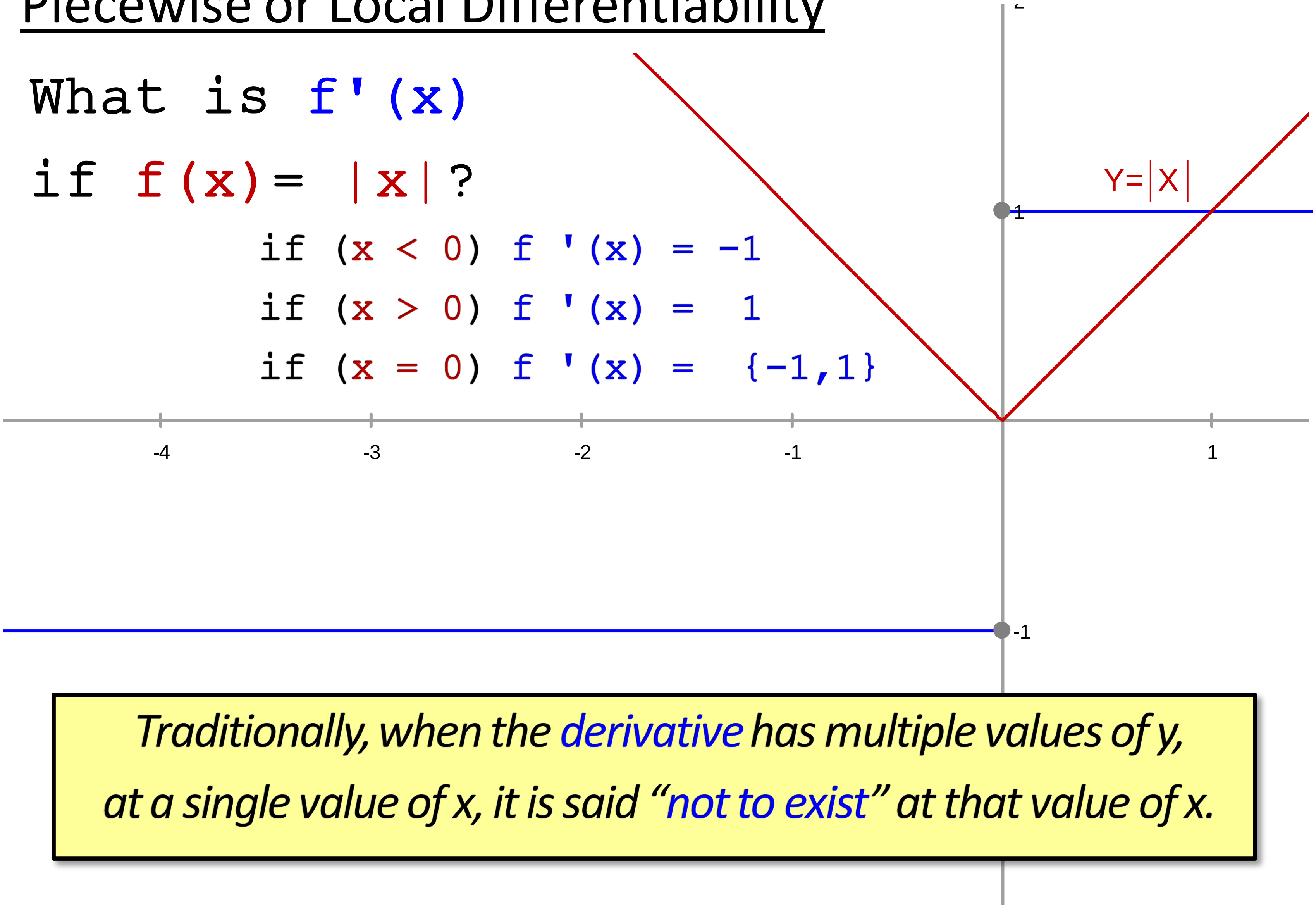
Piecewise or Local Differentiability

Lecture11-PiecewiseDifferentiable.gx

What is f'(x)

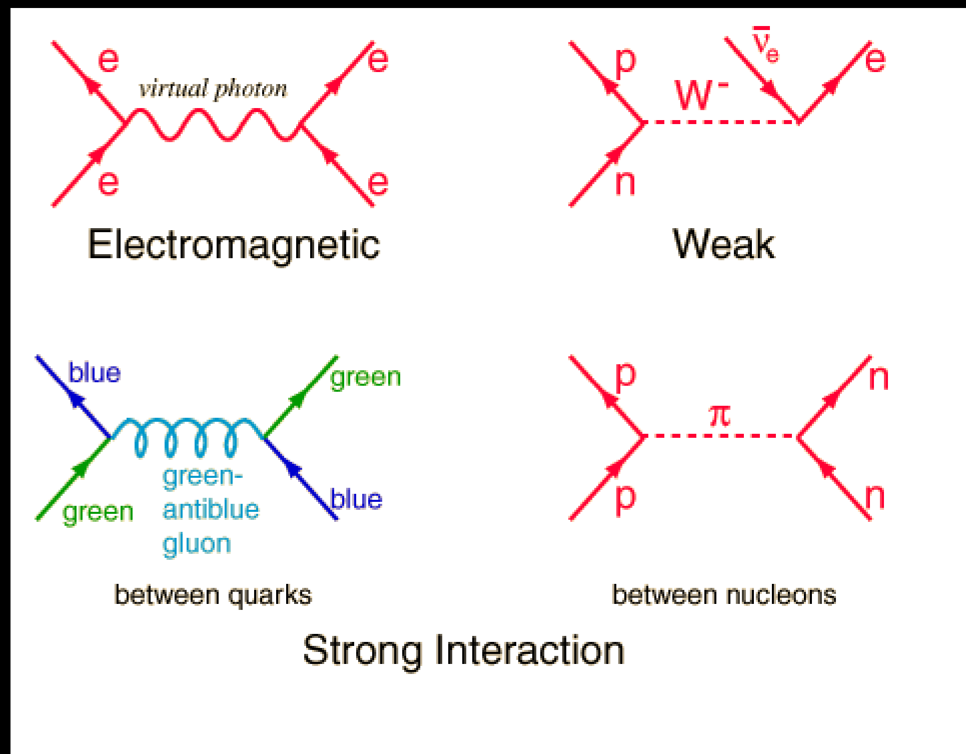
if f(x)= |x| ?

```
if (x < 0) f '(x) = −1
if (x > 0) f '(x) =  1
if (x = 0) f '(x) =  {−1,1}
```



Traditionally, when the derivative has multiple values of y, at a single value of x, it is said "not to exist" at that value of x.

Feynman Diagrams



End