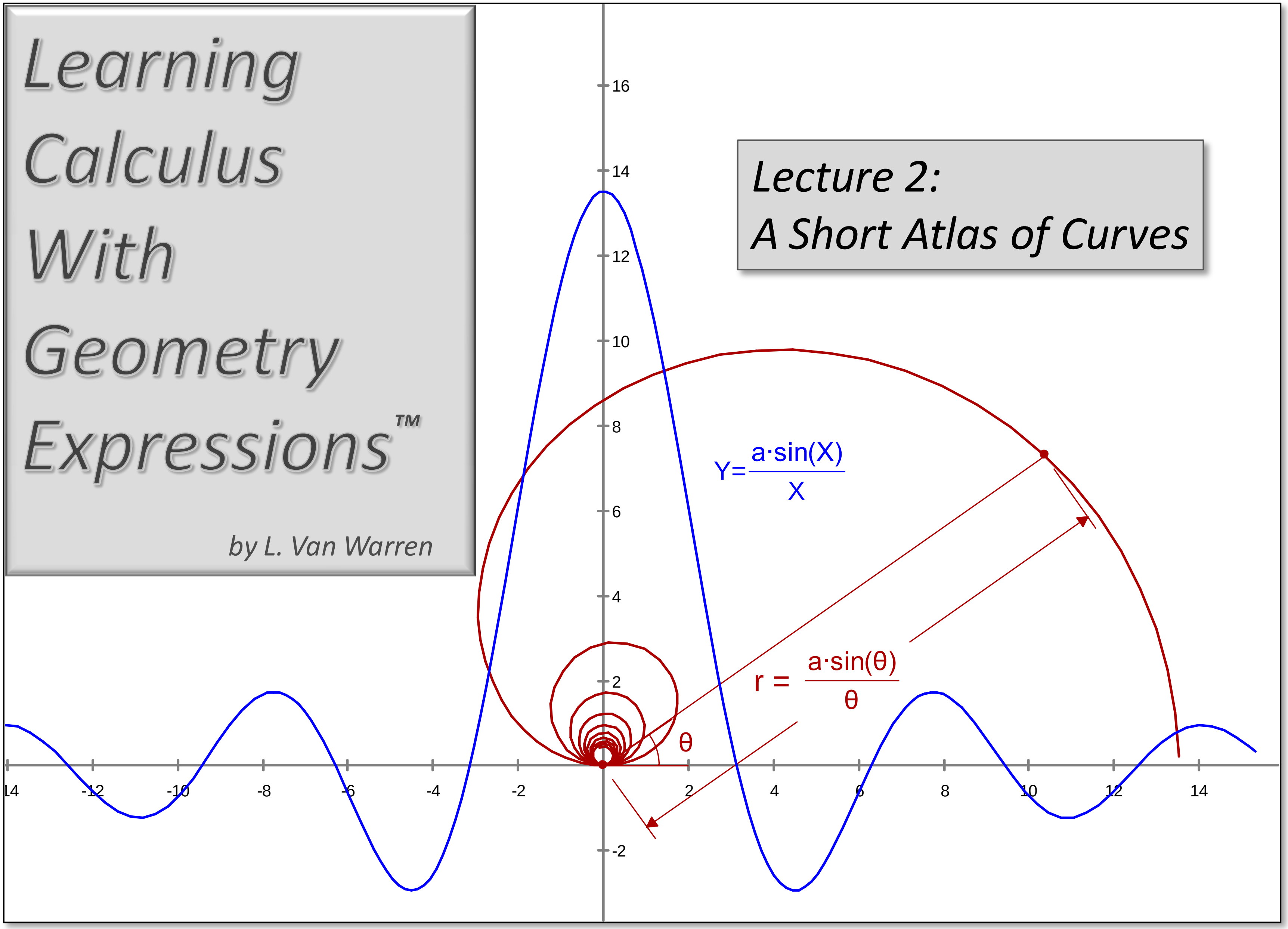
Learning Calculus With Geometry Expressions™

by L. Van Warren

Lecture 2: A Short Atlas of Curves



Chapter 1: Functions and Equations

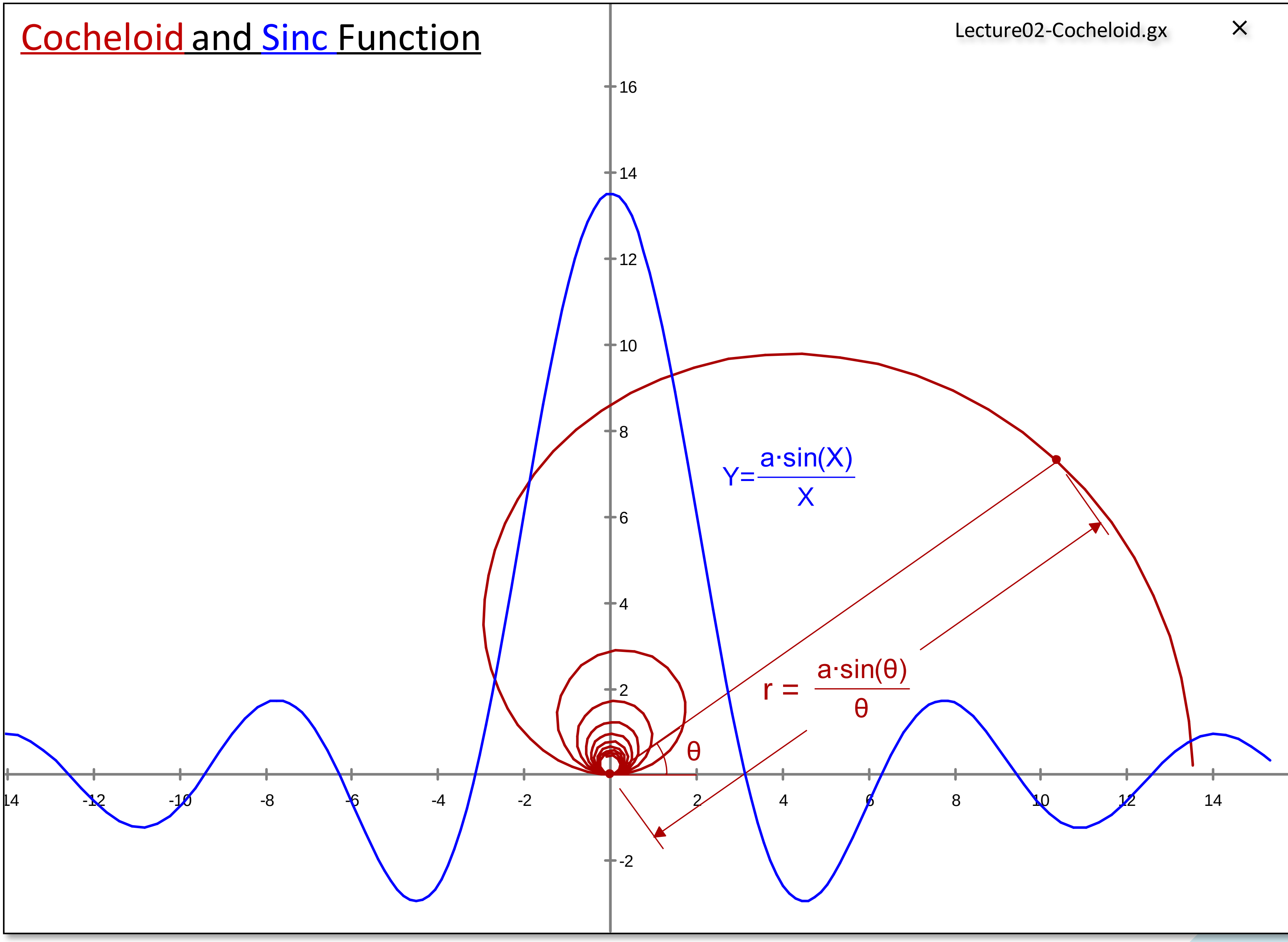
LECTURE	TOPIC
0	GEOMETRY EXPRESSIONS™ WARM-UP
1	EXPLICIT, IMPLICIT AND PARAMETRIC EQUATIONS
2	**A SHORT ATLAS OF CURVES**
3	SYSTEMS OF EQUATIONS
4	INVERTIBILITY, UNIQUENESS AND CLOSURE

Calculus Inspiration

Carolyn Shoemaker

Astronomer

- Began astronomy career at 50
- Discovered 32 Comets
  Record for a Single Person
- Discovered 800 Asteroids
- Co-discoverer of Shoemaker-Levy 9
- NASA Exceptional
  Scientific Achievement Medal

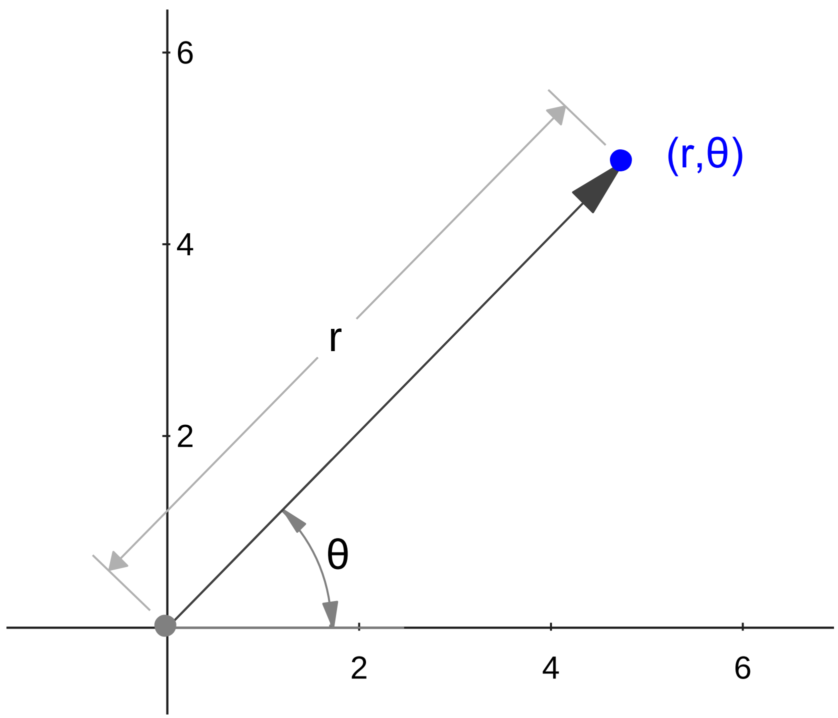

Cocheloid and Sinc Function
Lecture02-Cocheloid.gx
×
16
14
12
10
8
6
4
2
-2
-14
-12
-10
-8
-6
-4
-2
2
4
6
8
10
12
14
Y= a·sin(X) / X
r = a·sin(θ) / θ
θ

CARTESIAN VS. POLAR COORDINATE SYSTEMS

Up to now we have worked in a Cartesian Coordinate System, a regular grid, in the x and y directions. We can also work in a complementary coordinate system called a Polar Coordinate System or just Polar Coordinates for short. A point in Polar Coordinates is not located by its (x, y) point pair, but rather by its radial distance r out, and its counter clockwise rotation, θ from a horizontal axis. In Cartesian Coordinates the units of the point pair are often the same for x and y, feet, meters, etc. In Polar Coordinates, the radius r has units of distance, and θ has ***angular units*** of measure. In Calculus, radians are ***always*** used instead of degrees so that the relationship between arc length s, and angular measure θ, are preserved as in:

$$s = r \cdot \theta$$

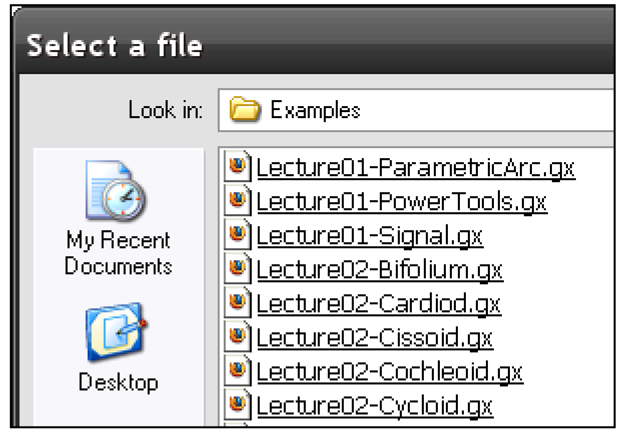
CARTESIAN VS. POLAR COORDINATE SYSTEMS



Plotting the same function in Polar Coordinates looks much different because instead of going over and up, in x and y, we are going out a distance r, and rotating by an angle θ.

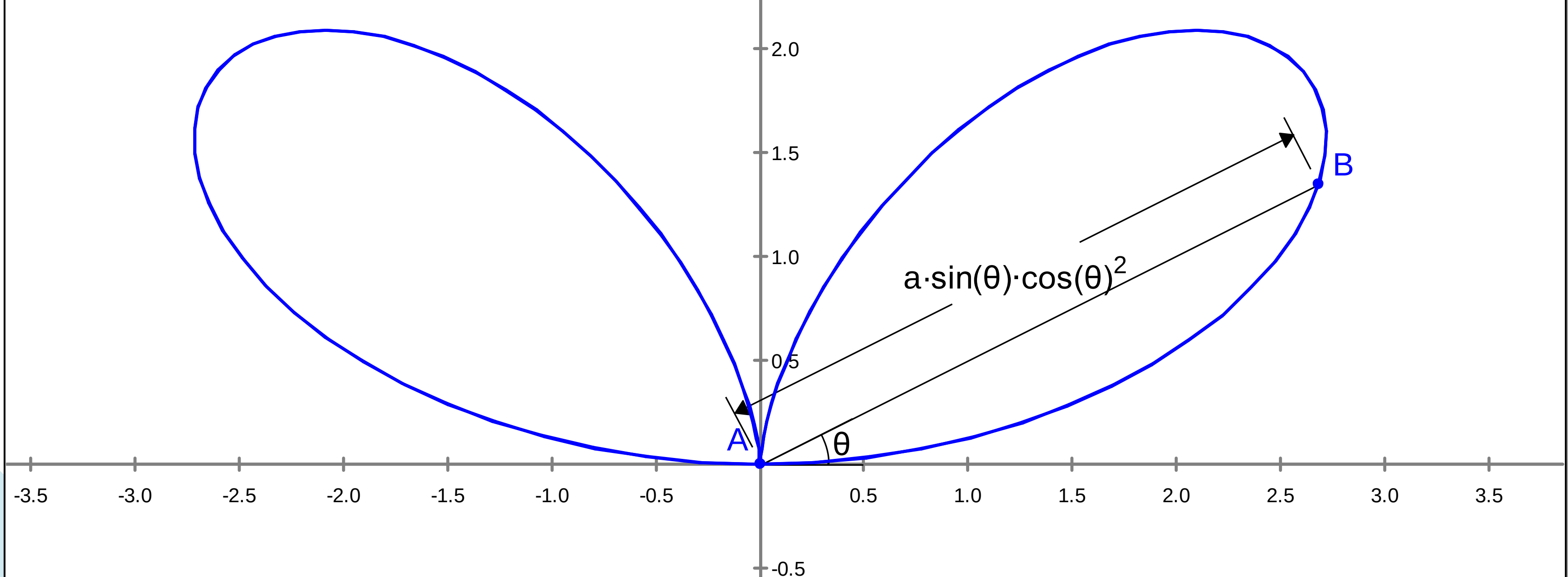
Bifolium Function

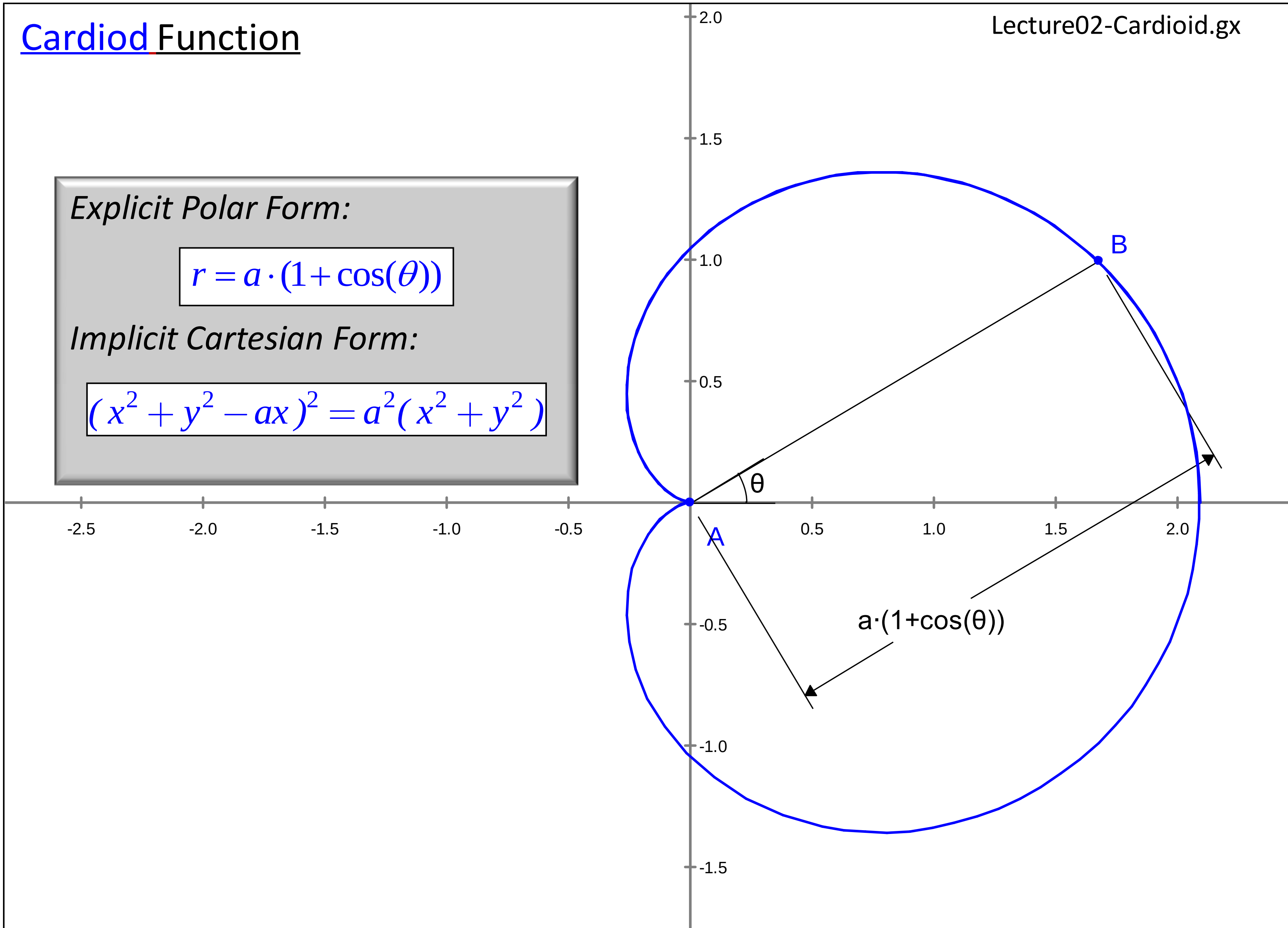
File→Open→Lecture02-Bifolium.gx



Exercises

1) Open Geometry Expressions™.
2) Open this example.
3) Vary parameter **a** in Variables Dialog.
4) Find an implicit expression for this curve.





Cardiod Function
Lecture02-Cardioid.gx
Explicit Polar Form:
$r = a \cdot (1 + \cos(\theta))$
Implicit Cartesian Form:
$(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2)$
B
A
θ
a·(1+cos(θ))
2.0
1.5
1.0
0.5
-0.5
-1.0
-1.5
-2.5
-2.0
-1.5
-1.0
-0.5
0.5
1.0
1.5
2.0

Cissoid of Diocles Function

Lecture02-Cissoid.gx

Explicit Polar Form:

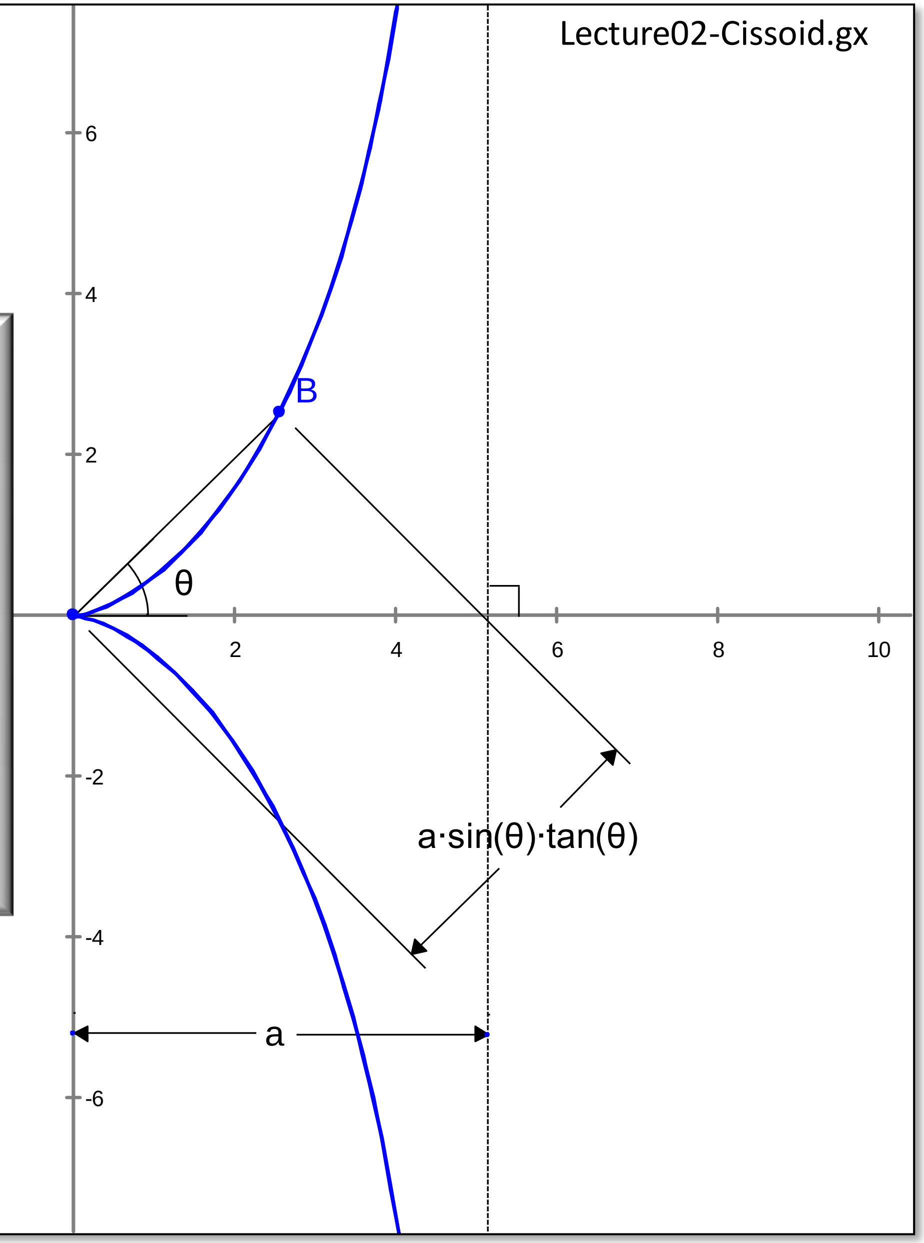
$$r = a \cdot \sin(\theta) \cdot \tan(\theta)$$

Implicit Cartesian Form:

$$y^2(a - x) = x^3$$

Exercise:

- *Vertical Asymptote at x = a*
- *Cusp at origin.*



Sine and Cosine Functions

Lecture02-SinCos.gx

Explicit Polar Form

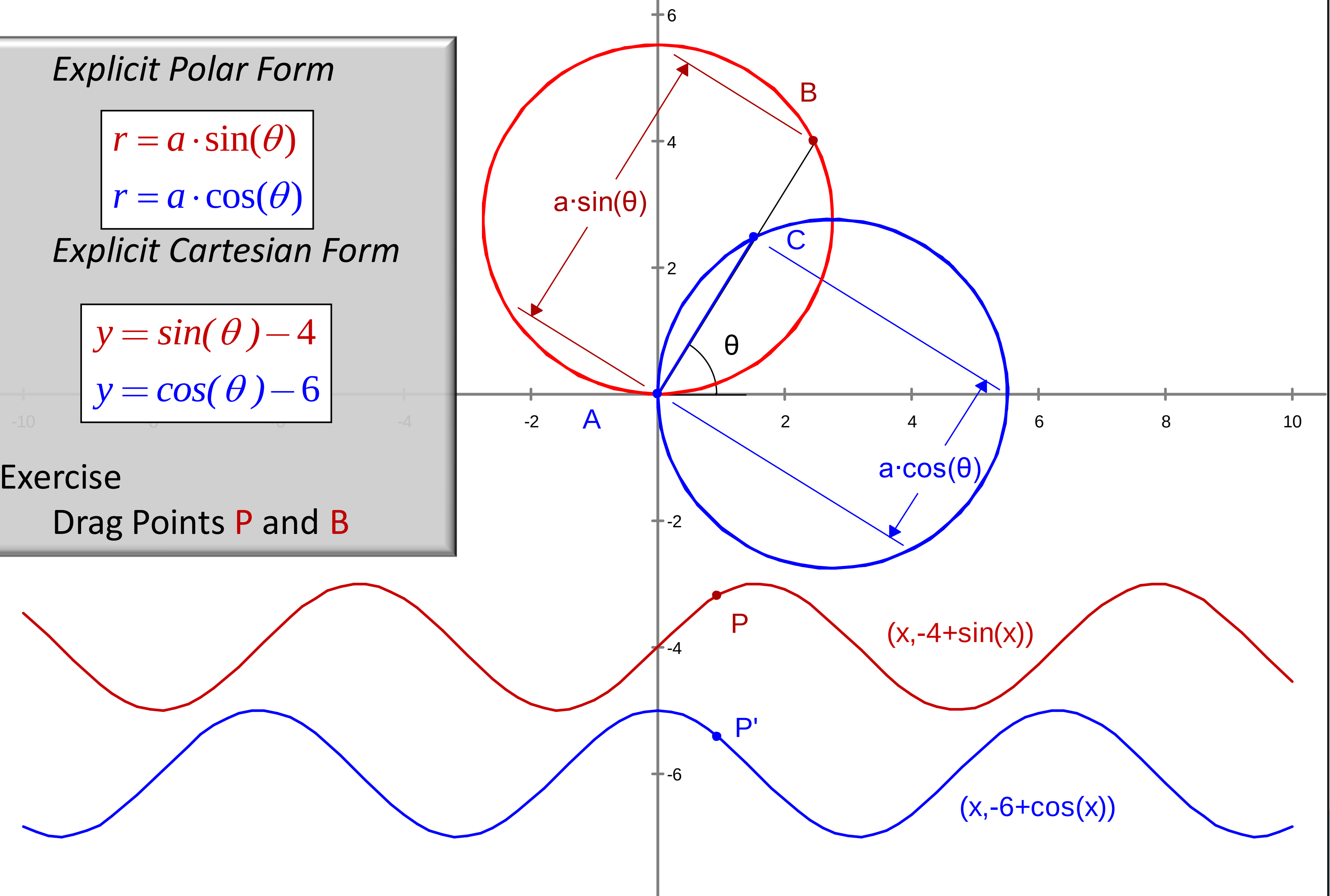
$$r = a \cdot \sin(\theta)$$
$$r = a \cdot \cos(\theta)$$

Explicit Cartesian Form

$$y = sin(\theta) - 4$$
$$y = cos(\theta) - 6$$

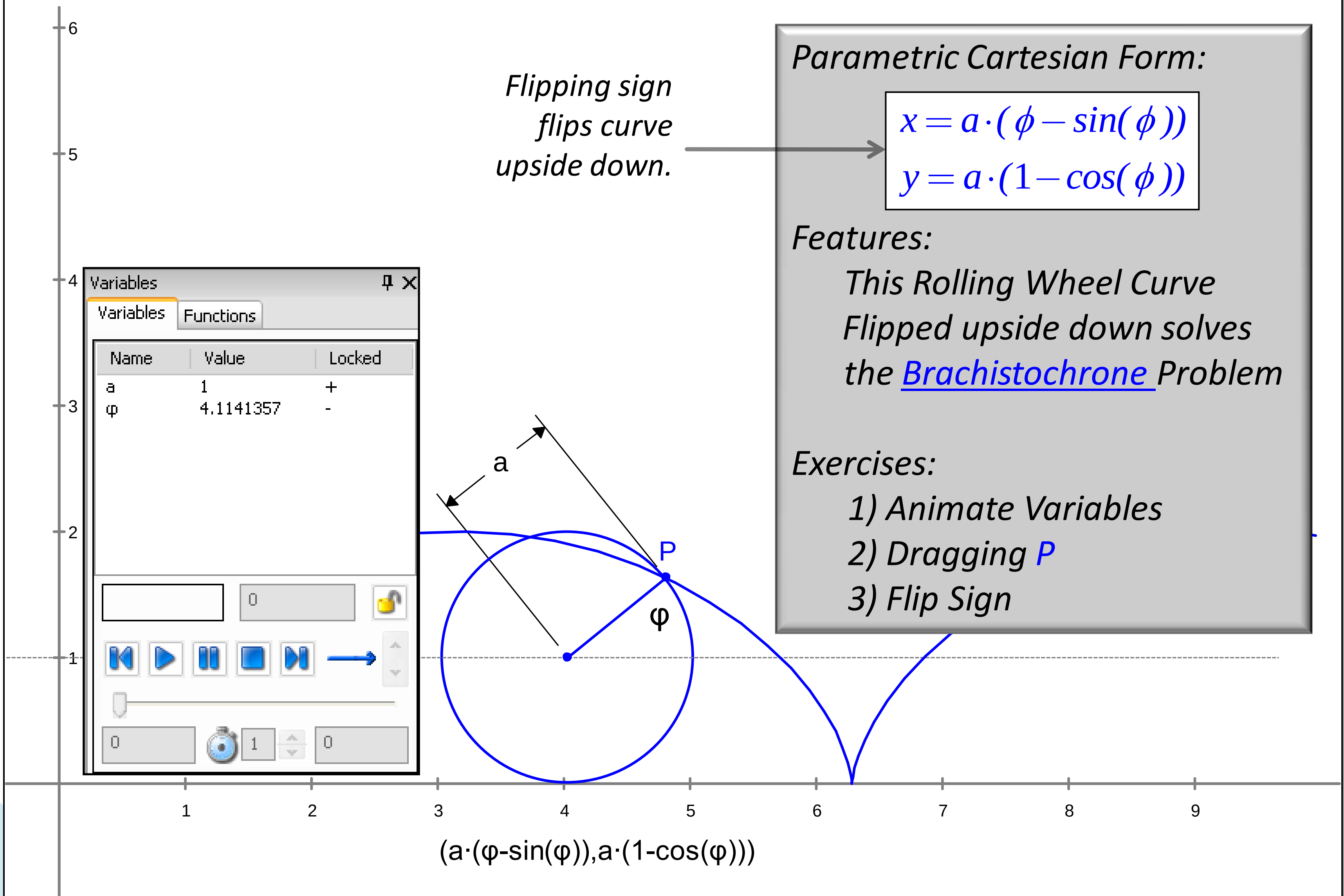
Exercise

Drag Points P and B



Cycloid Function

Lecture02-Cycloid.gx



Flipping sign flips curve upside down.

Parametric Cartesian Form:

$$x = a\cdot(\phi - \sin(\phi))$$
$$y = a\cdot(1 - \cos(\phi))$$

Features:

This Rolling Wheel Curve Flipped upside down solves the Brachistochrone Problem

Exercises:

1. *Animate Variables*
2. *Dragging P*
3. *Flip Sign*

Hyperbolic Functions

Lecture02-SinhCoshTanh.gx

Explicit Form:

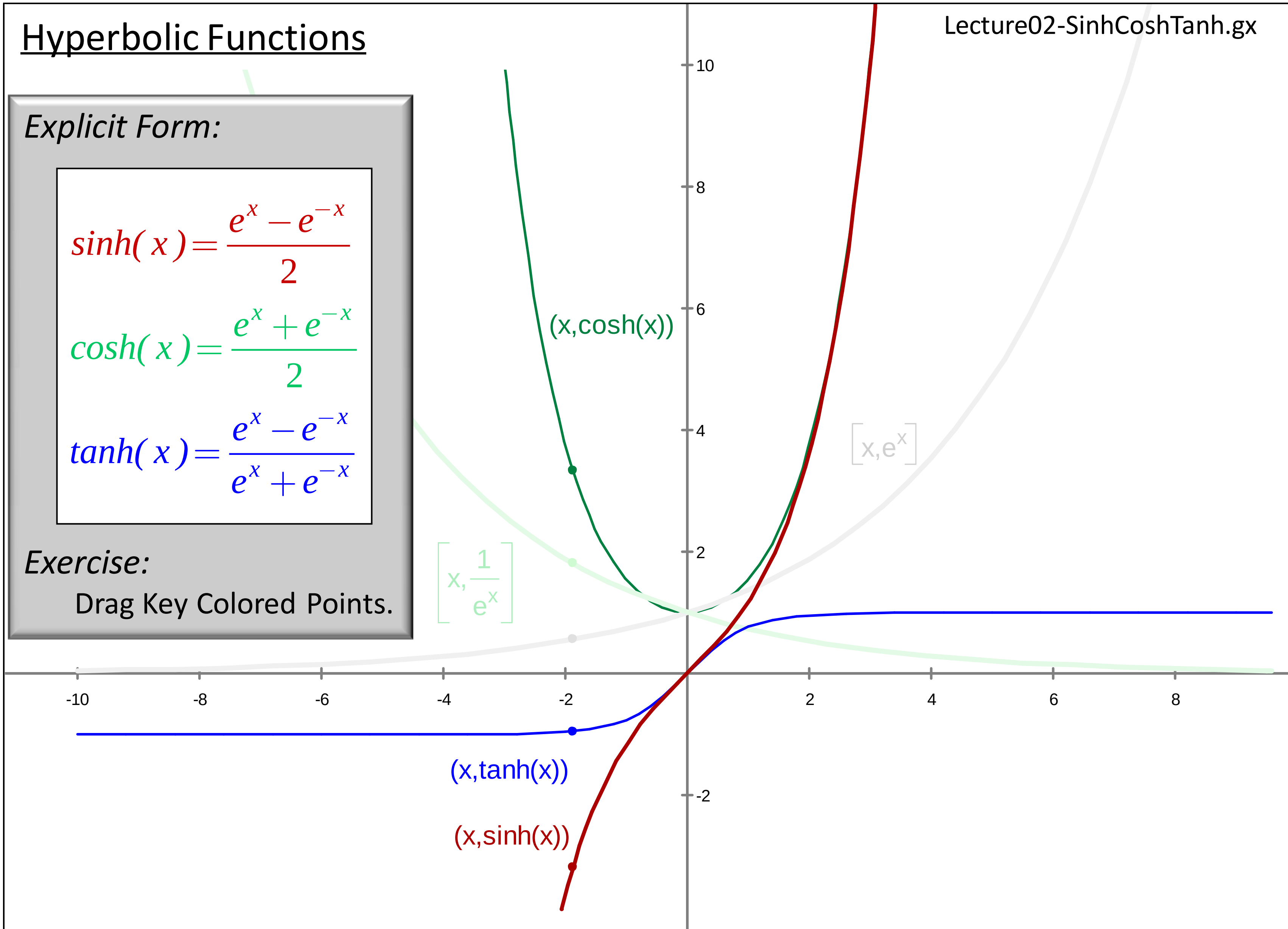
$$sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Exercise:

Drag Key Colored Points.



Epicycloid: Wheel on Wheel

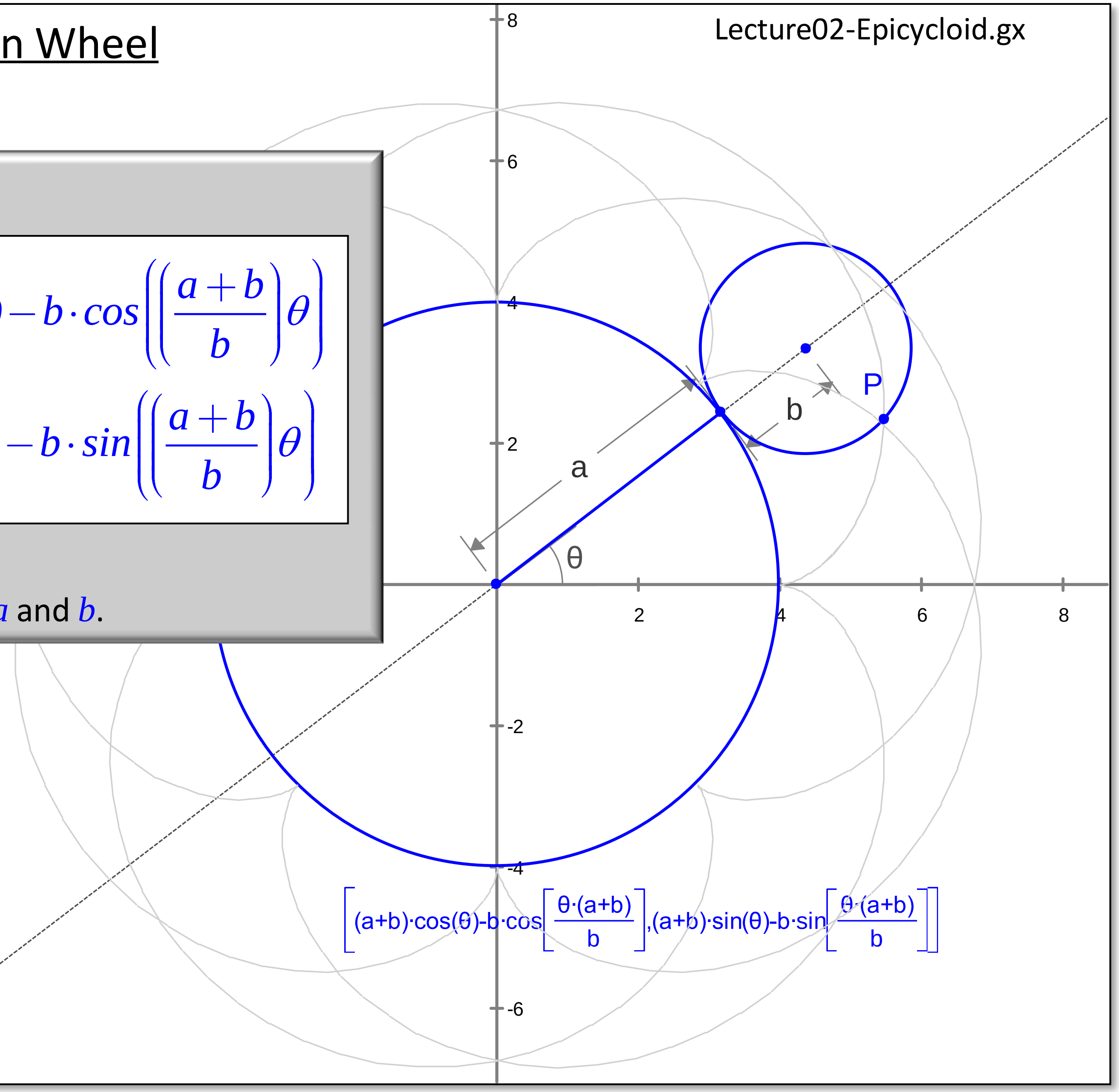
Explicit Cartesian Form:

$$x=(a+b)\cos(\theta)-b\cdot\cos\left(\left(\frac{a+b}{b}\right)\theta\right)$$

$$y=(a+b)\sin(\theta)-b\cdot\sin\left(\left(\frac{a+b}{b}\right)\theta\right)$$

Exercise:

Animate Parameters a and b.



The Ellipse

Lecture02-Ellipse.gx

Parametric Cartesian Form:

$$x = a \cdot cos(\theta)$$
$$y = b \cdot sin(\theta)$$

Exercise:

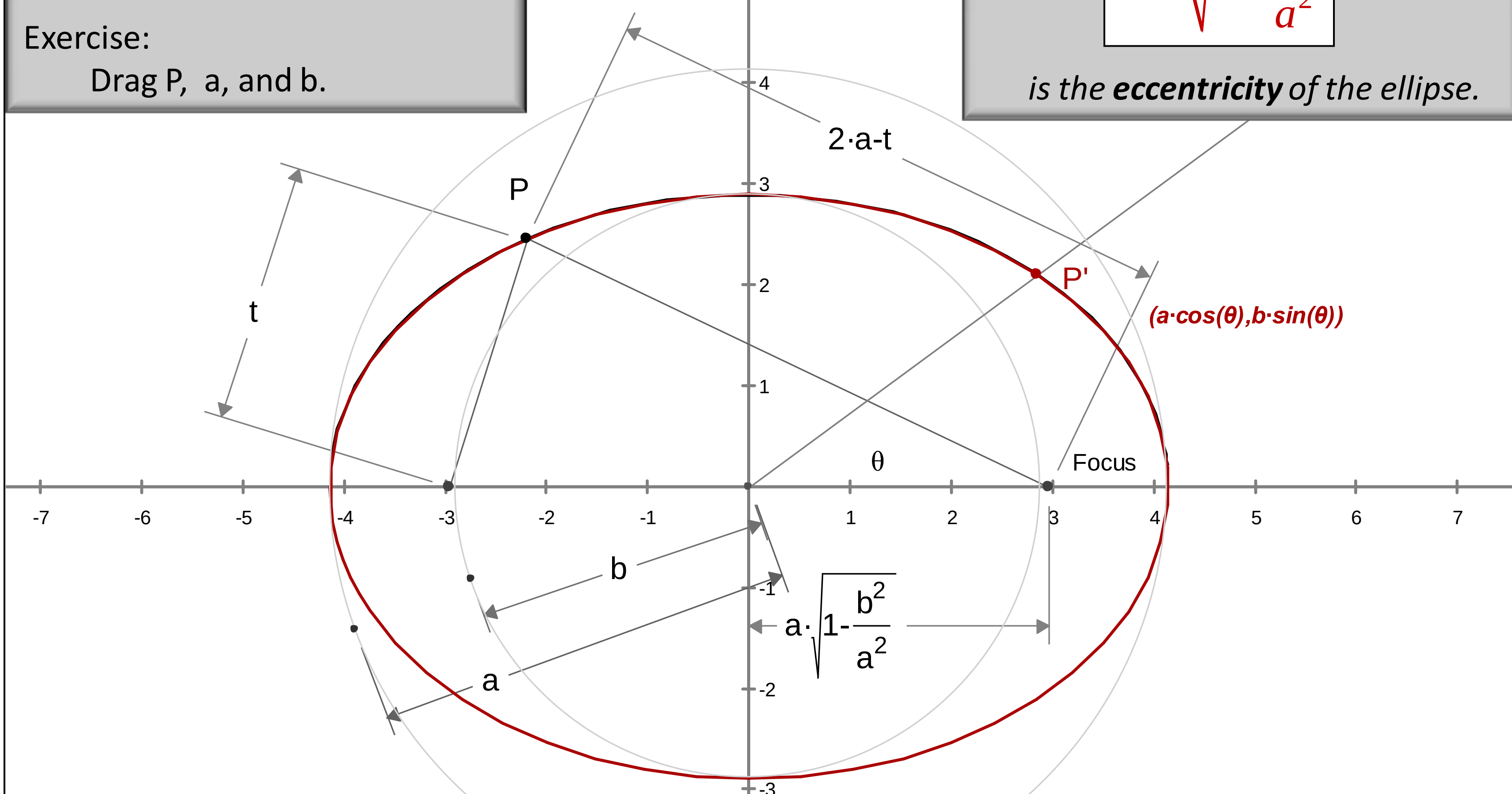
Drag P, a, and b.

Eccentricity :

The quantity:

$$\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}$$

*is the **eccentricity** of the ellipse.*



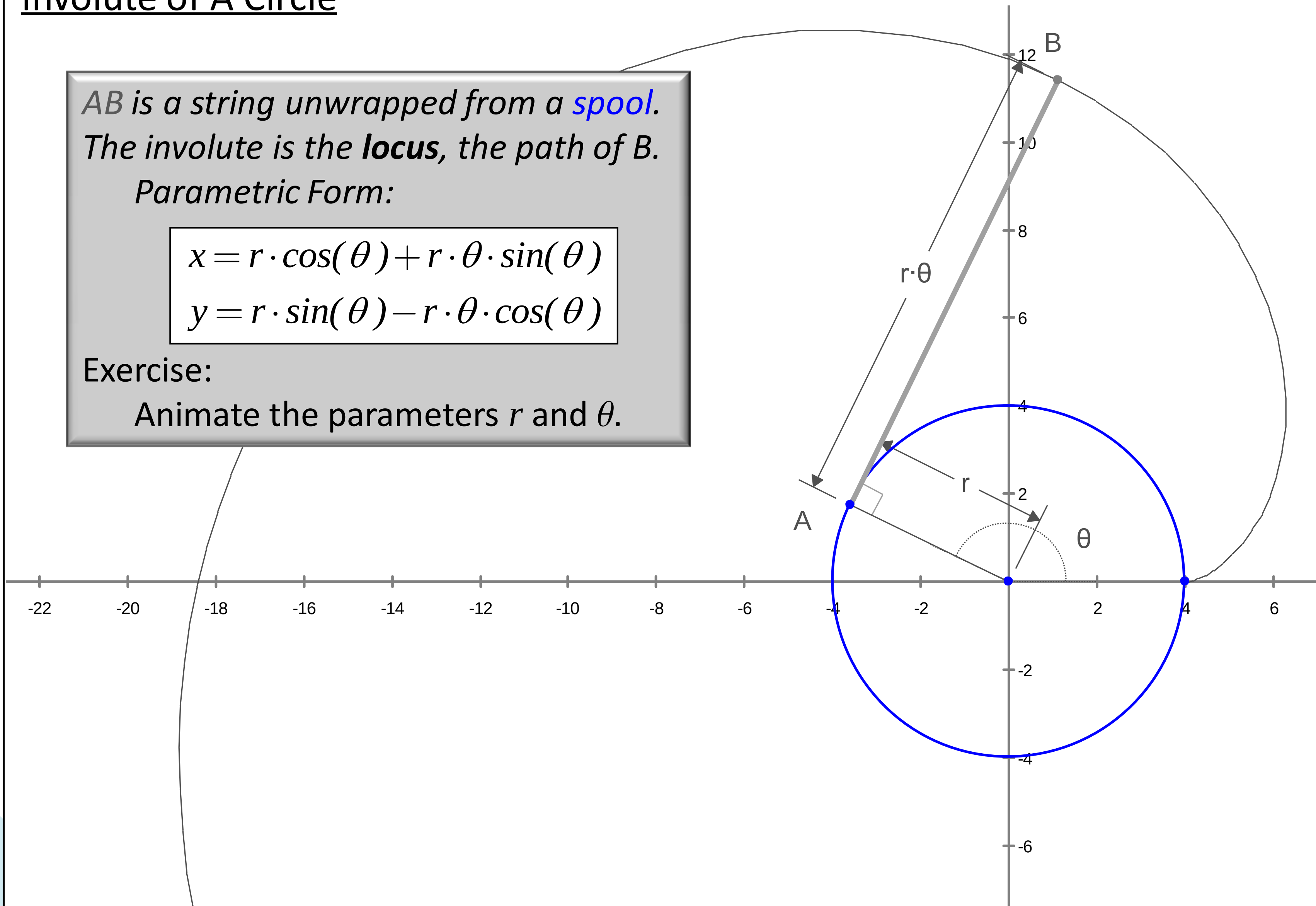
Involute of A Circle

Lecture02-InvoluteOfCircle.gx

AB is a string unwrapped from a spool.
*The involute is the **locus**, the path of B.*
Parametric Form:

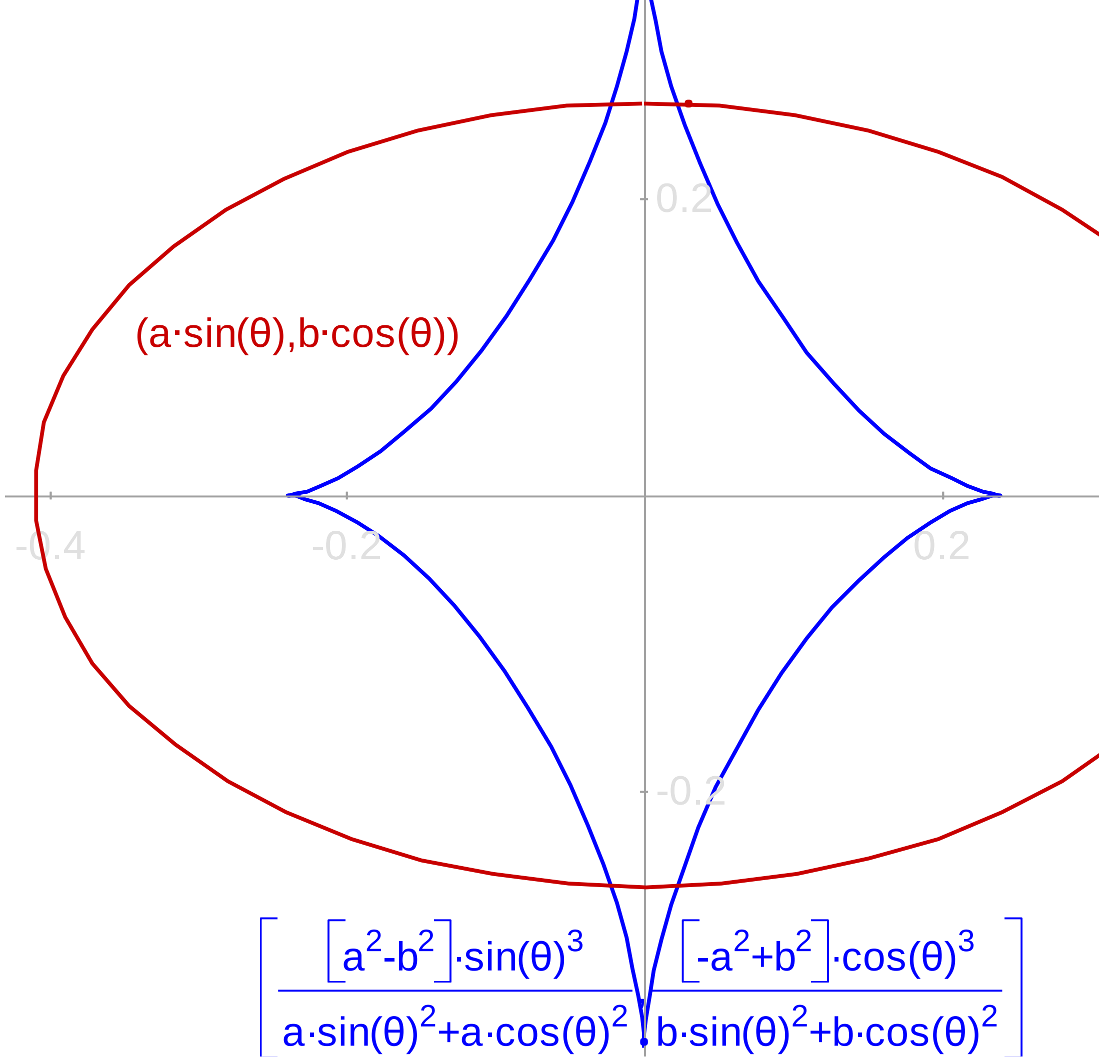
$$x = r \cdot \cos(\theta) + r \cdot \theta \cdot \sin(\theta)$$
$$y = r \cdot \sin(\theta) - r \cdot \theta \cdot \cos(\theta)$$

Exercise:
Animate the parameters r and θ.



Evolute of an Ellipse

Lecture02-EvoluteOfEllipse.gx



Evolutes and Involutes are inverses operations.

Ellipse Parametric Form:

$$x = a \cdot sin(\theta)$$
$$y = b \cdot cos(\theta)$$

Exercise:

1) Drag the point (a,b).
2) What is the evolute of a circle?

Natural Logarithm Spiral

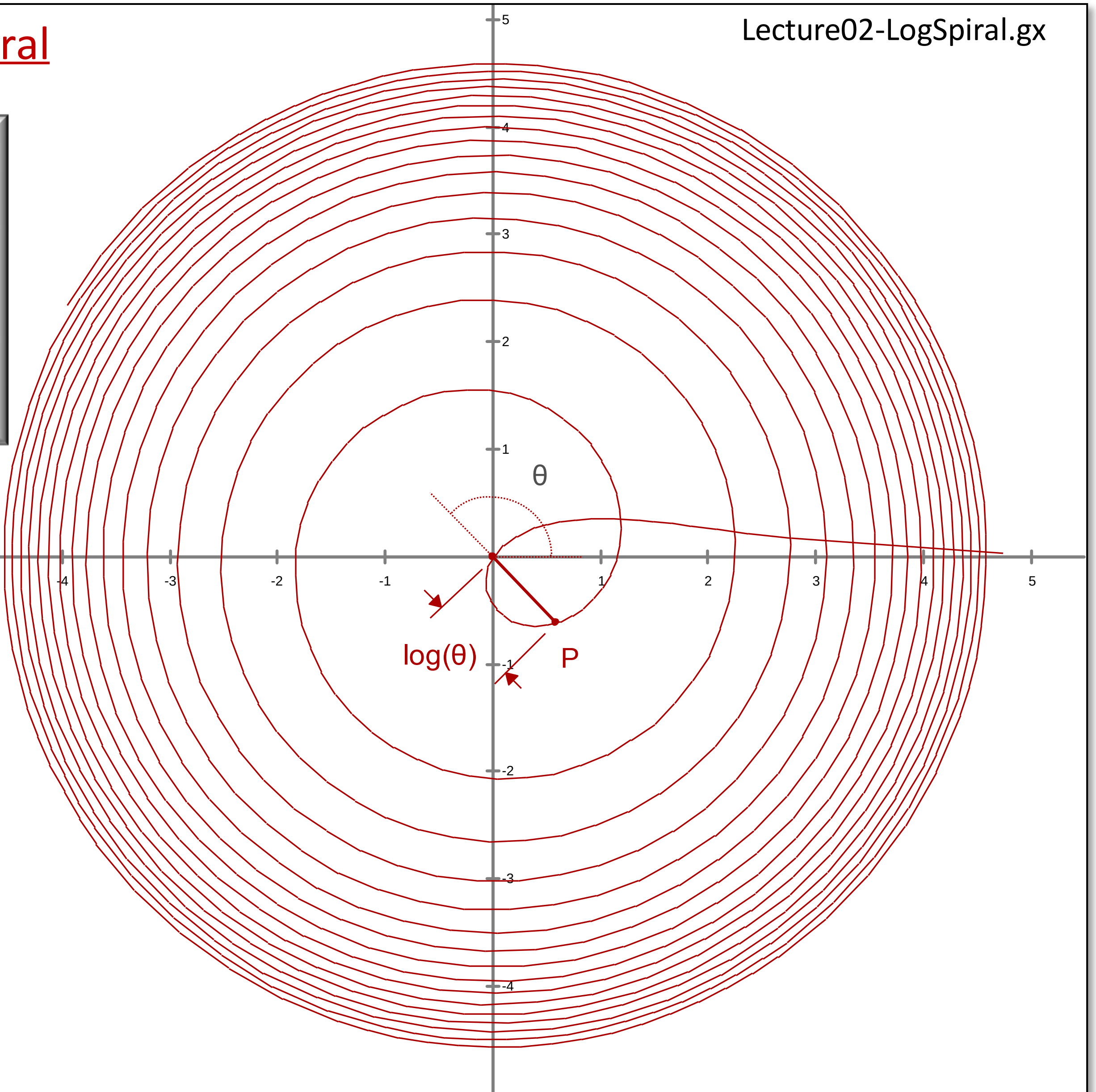
Lecture02-LogSpiral.gx

Explicit Polar Form:

$$r = \log_e(\theta)$$

Exercise:

Animate θ.

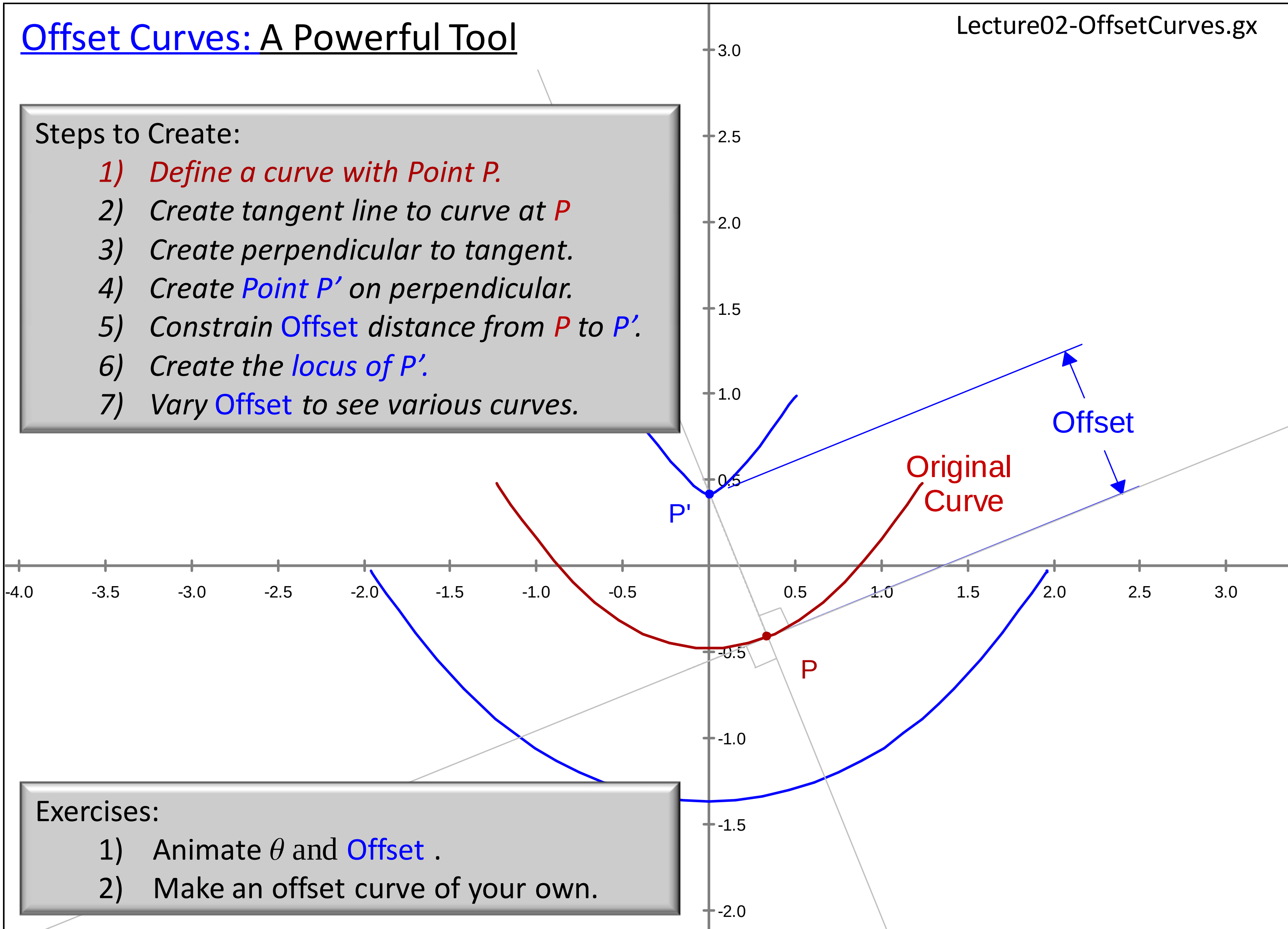


Offset Curves: A Powerful Tool

Lecture02-OffsetCurves.gx

Steps to Create:

1) *Define a curve with Point P.*
2) *Create tangent line to curve at P*
3) *Create perpendicular to tangent.*
4) *Create Point P' on perpendicular.*
5) *Constrain Offset distance from P to P'.*
6) *Create the locus of P'.*
7) *Vary Offset to see various curves.*



Exercises:

1) Animate θ and Offset .
2) Make an offset curve of your own.

End