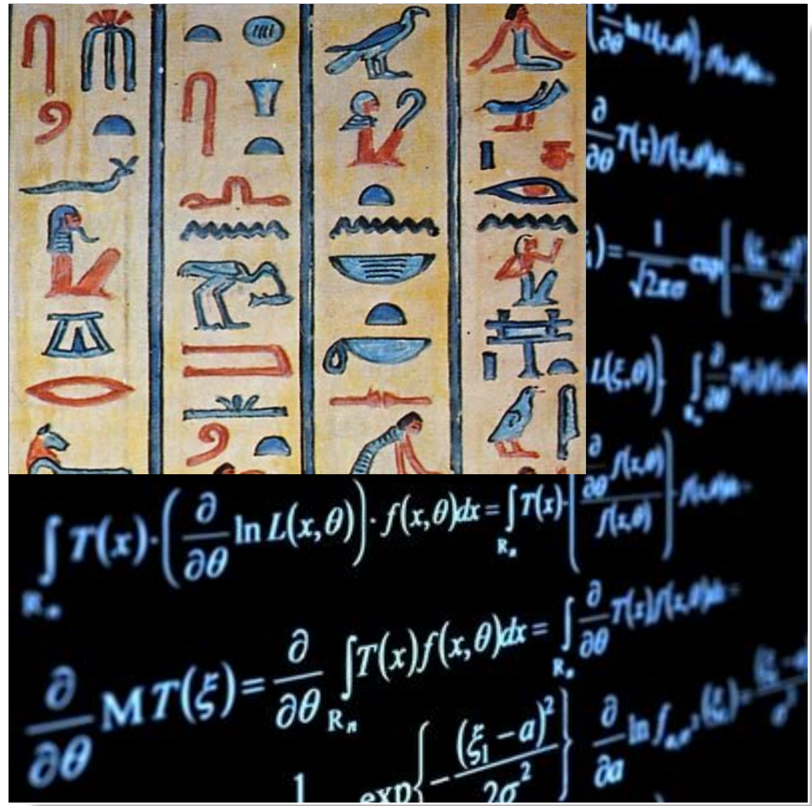
Learning Calculus With Geometry Expressions™

by L. Van Warren



- torsors

Lecture 25:
Integration By Parts

Chapter 6: Integration Techniques

Lecture	Topic
23	Integration: Cartesian and Polar
24	Integration by Substitution
25	**Integration by Parts**
26	Area Between Curves

Calculus Inspiration

Julia Robinson

PhD. Berkeley 1948
University of California
Thesis: Definability and Decision
Problems in Arithmetic

- Laid Foundation for Hilbert's 10th Problem.
- Diophantine Equations & Computability
- First Woman President AMS

Alfred Tarski (26)

Stanislaw Lesniewski

Kazimierz Twardowski (4)

Robert Zimmerman

Bernard Bolzano (2)

Examples of Diophantine Equations:

Lecture25-Diophantine.wxm

In Diophantine equations, a and b are given, while i, j, and k are unknowns.

$$a \cdot i + b \cdot j = 1$$ *Linear Diophantine Equation*

$$7 \cdot i + 18 \cdot j = 208$$

$$a^i + b^i = c^i$$ *Pythagorean for $i = 2$*

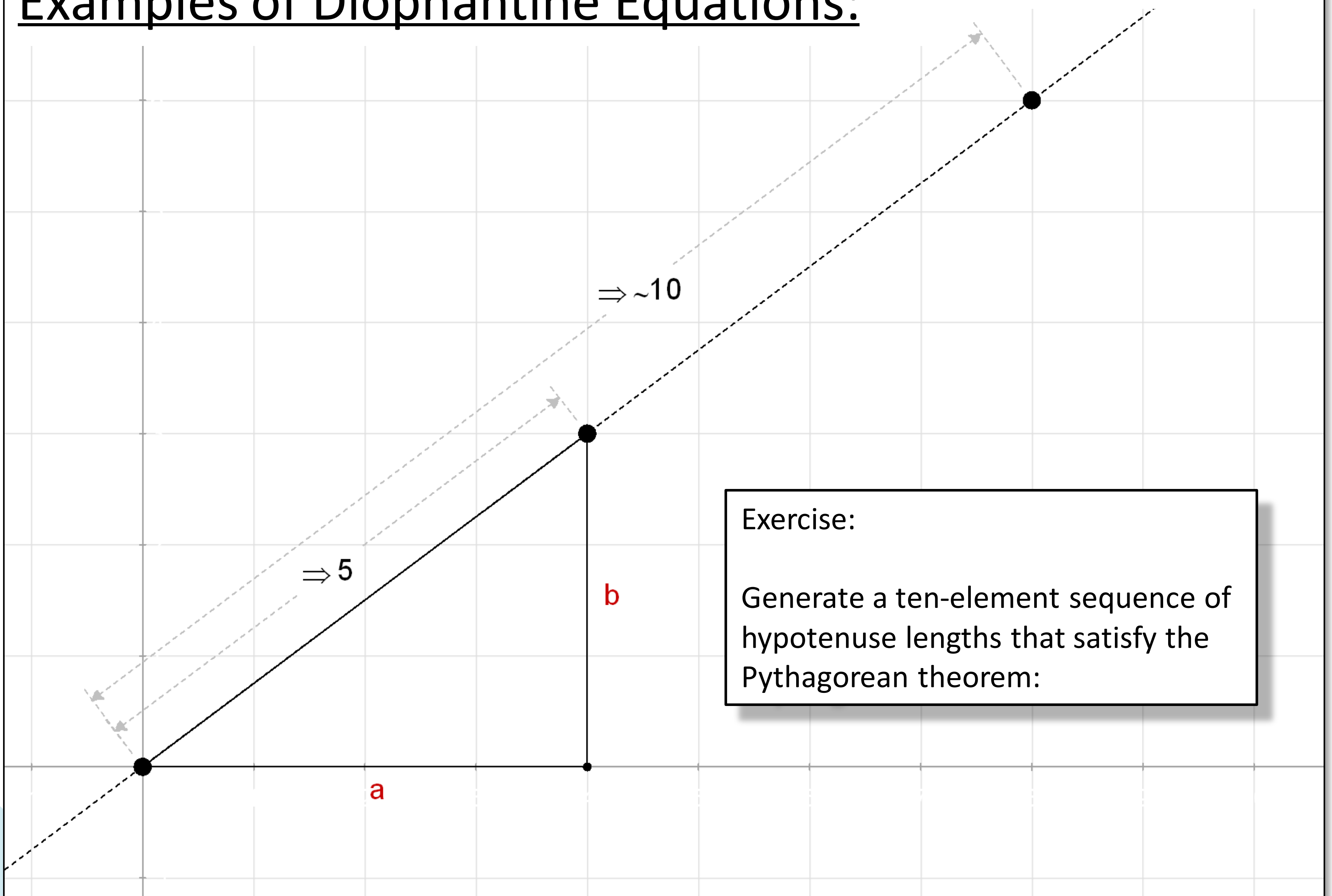
Unsolveable for $i > 2$

```
create_list(a*i+b*j=1,i,-1,1,j,-1,1);
[-b-a=1, -a=1, b-a=1, -b=1, 0=1, b=1, a-b=1, a=1, b+a=1]

create_list(x^2+y^2=z^2,x,3,5,y,3,5);
[18=z^2, 25=z^2, 34=z^2, 25=z^2, 32=z^2, 41=z^2, 34=z^2, 41=z^2, 50=z^2]
```

Examples of Diophantine Equations:

Lecture25-Diophantine.gx



Exercise:

Generate a ten-element sequence of hypotenuse lengths that satisfy the Pythagorean theorem:

Integration By Parts 101:

A powerful symbolic trick:

$$\int u \, dv = u \cdot v - \int v \, du$$

Integration By Parts 101:

Consider integration by parts on the simplest integral.

$$\int dx \quad = ?$$

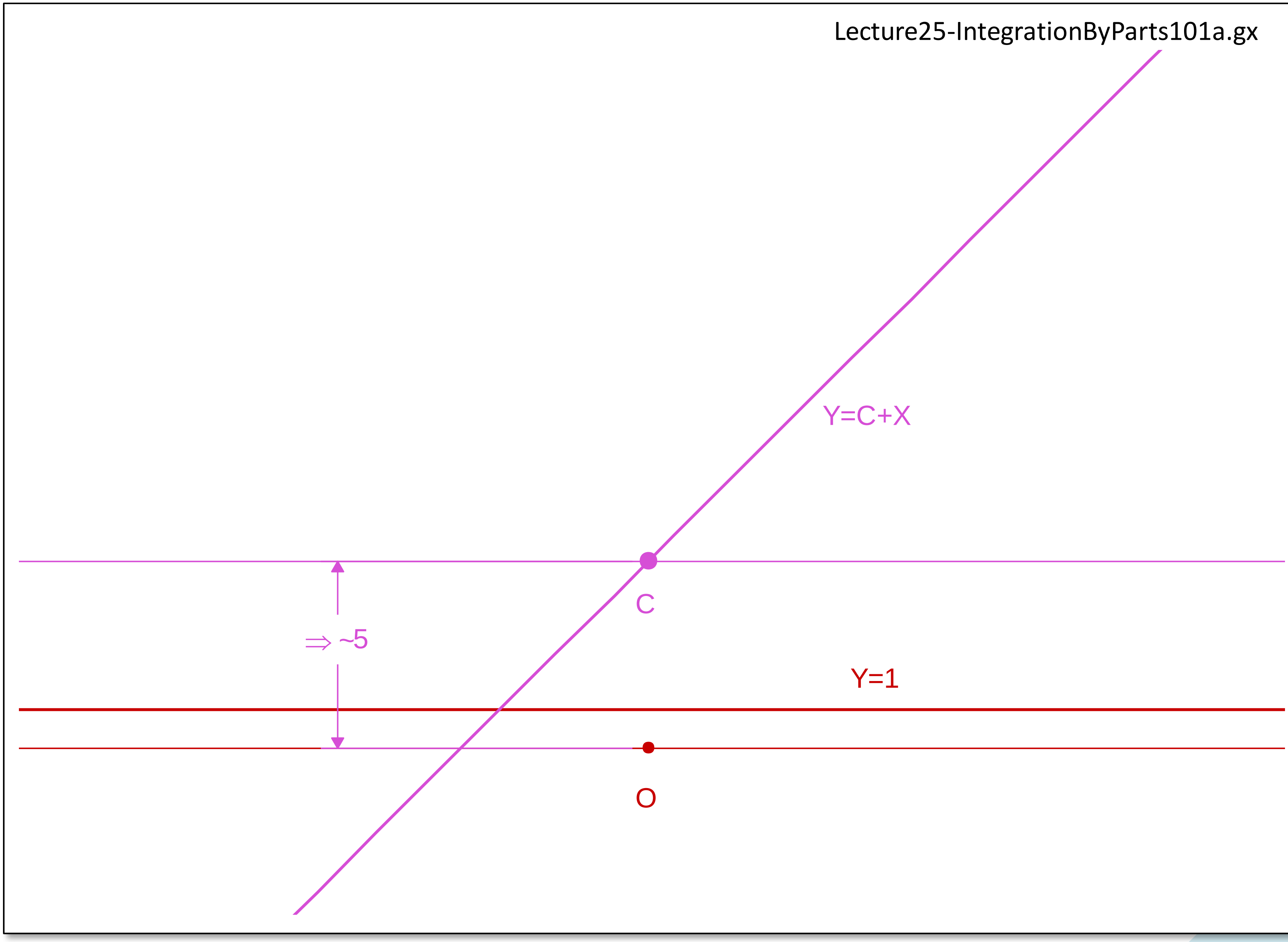
$$\# \text{ Let} \quad u = 1 \quad \text{then} \quad du = 0 \cdot dx$$

$$\# \text{ Let} \quad dv = dx \quad \text{then} \quad v = x$$

$$\int u \quad dv = u \cdot v - \int v \quad du \qquad \# \text{ The Rule}$$

$$\int 1 \cdot dx \quad = 1 \cdot x - 0 \int x \; dx \qquad \# \text{ Substitution}$$

$$\int 1 \cdot dx \quad = \quad x + C \qquad \# \text{ Result}$$


Lecture25-IntegrationByParts101a.gx
Y=C+X
C
⇒ ~5
Y=1
O

Integration By Parts 101:

Consider integration by parts on a non-trivial example.

$$\int x \cdot \cos(x)\, dx \quad = \; ?$$

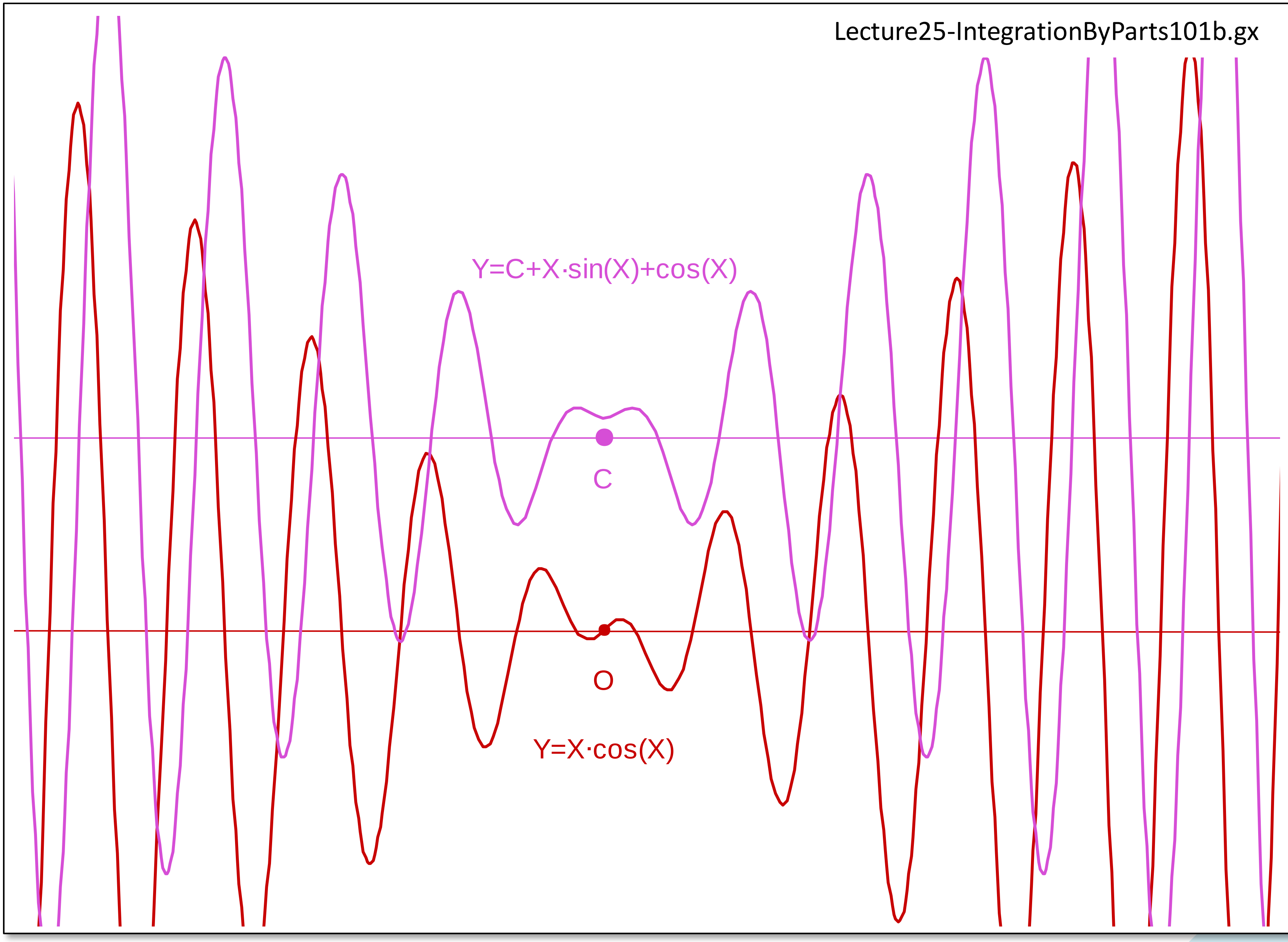
$$\# \text{ Let } \quad u = x \qquad \text{then} \quad du = dx$$

$$\# \text{ Let } \quad dv = \cos(x)\, dx \quad \text{then} \quad v = \sin(x)$$

$$\int u \; dv = u \cdot v - \int v \; du \qquad \# \text{ The Rule}$$

$$\int x \cdot \cos(x)\, dx \quad = \; x \cdot \sin(x) + \cos(x) + C \quad \# \text{ Substitution}$$

Exercise: Why could we omit the constant of integration in the dv=cos(x)dx step?


Lecture25-IntegrationByParts101b.gx
Y=C+X·sin(X)+cos(X)
C
O
Y=X·cos(X)

Integration By Parts 201:

A more complex example of:

$$\int u\,dv = u \cdot v - \int v\,du$$

Integration By Parts 201:

Consider integration by parts on a non-trivial example.

$$\int \frac{(1-x)}{(1+x)} \cdot dx \quad = ?$$

$$\# \text{ Let} \quad u = 1 - x \quad \text{then} \quad du = -dx$$

$$\# \text{ Let} \quad dv = \frac{dx}{1+x} \quad \text{then} \quad v = log(1+x)$$

$$\int u \quad dv = u \cdot v - \int v \quad du \qquad \# \text{ The Rule}$$

$$\int \frac{(1-x)}{(1+x)} \cdot dx = (1-x) \cdot log(1+x) + \int log(1+x) dx$$

$$= 2 \cdot log(1+x) - x + C$$

Again Using Maxima™:

Lecture25-IntegrationByParts201a.gx

```
u(x):=(1-x);              du_dx(x):=diff(u(x),x)$
du_dx(x);
u(x):=1-x
-1
```

In the first section we process u.

```
dv_dx(x):=1/(1+x);    v(x):=integrate(dv_dx(x),x)$
v(x);
```

$$dv\_dx(x) := \frac{1}{1+x}$$

```
log(x+1)
```

In the second section we process dv.

Exercise: Explain why our result and Maxima's check differ.

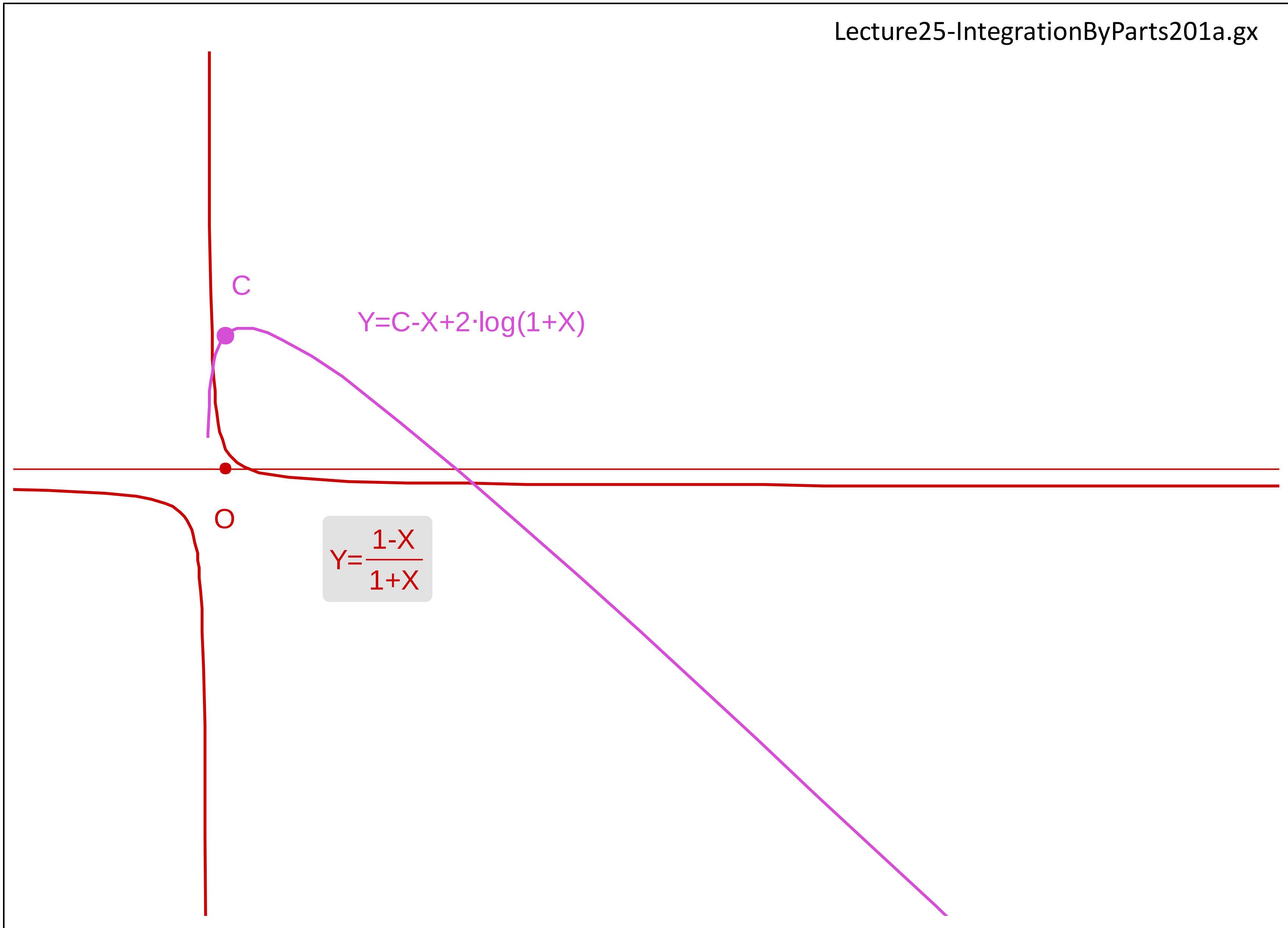
```
u(x)*v(x);
(1-x) log(x+1)

ratsimp(u(x)*v(x)-integrate(v(x)*du_dx(x),x));
2 log(x+1)-x-1
```

← *Our result.*

```
integrate((1-x)/(1+x),x);
2 log(x+1)-x
```

← *Let Maxima™ check our work.*



Lecture25-IntegrationByParts201a.gx
C
Y=C-X+2·log(1+X)
O
Y=(1-X)/(1+X)

Integration By Parts 301:

The LIATE rule is a heuristic – or rule of thumb, constructed by Herbert Kasube and guides the selection of *u* in:

$$\int u\,dv = u \cdot v - \int v\,du$$

The letters in LIATE stand for:

L: Log functions: $\log_a(x)$…
I: Inverse trig functions: asin(x), acos(x), atan(x)…
A: Algebraic functions: ax^n**…**
T: Trig functions: sin(x), cos(x), tan(x)….
E: Logarithmic functions: a^x, e^x…

The selection of *dv* is guided by the reverse acronym DETAIL, where the D stands for *dv*.

<u>Integration By Parts 301:</u>

Consider this example where we integrate twice by parts:

$$\int e^x \sin(x) \cdot dx \quad = \; ?$$

$$\# \text{ Per LIATE Let} \quad u = \sin(x) \quad \text{then} \quad du = \cos(x)dx$$

$$\# \text{ Let} \quad dv = e^x dx \quad \text{then} \quad v = e^x$$

$$\int e^x \sin(x) \cdot dx = e^x \sin(x) - \int e^x \cos(x)dx \quad \# \text{ Eq. A}$$

$$\# \text{ Per LIATE Let} \quad u = \cos(x) \quad \text{then} \quad du = -\sin(x)dx$$

$$\# \text{ Let} \quad dv = e^x dx \quad \text{then} \quad v = e^x$$

$$\int e^x \cos(x)dx = e^x \cos(x) + \int e^x \sin(x)dx \quad \# \text{ Eq. B}$$

Integration By Parts 301:

Now we add equation A to a rearranged equation B:

$$\int e^x \sin(x)dx = e^x \sin(x) - \int e^x \cos(x)dx \quad \# \; Eq. A$$

$$\int e^x \sin(x)dx = -e^x \cos(x) + \int e^x \cos(x)dx \quad \# \; Eq. B'$$

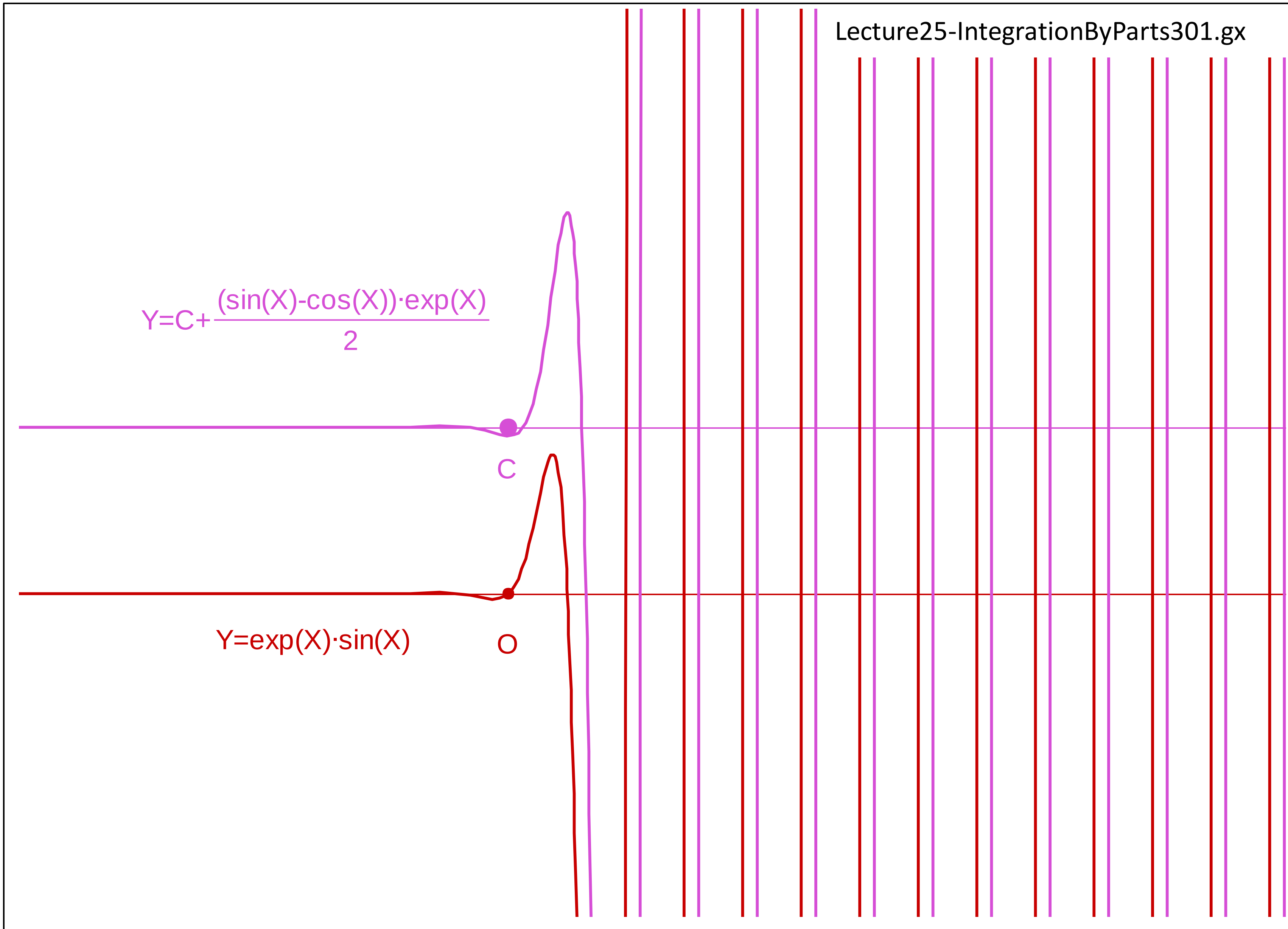
This produces the following equivalence relation:

$$2\int e^x \sin(x)dx = e^x(\sin(x) - \cos(x))$$

Which when rearranged provides our answer:

$$\int e^x \sin(x)dx = \frac{e^x(\sin(x) - \cos(x))}{2} + C$$

And we can't forget the constant of integration!


Lecture25-IntegrationByParts301.gx
Y=C+(sin(X)-cos(X))·exp(X)/2
C
Y=exp(X)·sin(X)
O

$$\int uv\, dw = uvw - \int uw\, dv - \int vw\, du$$

End