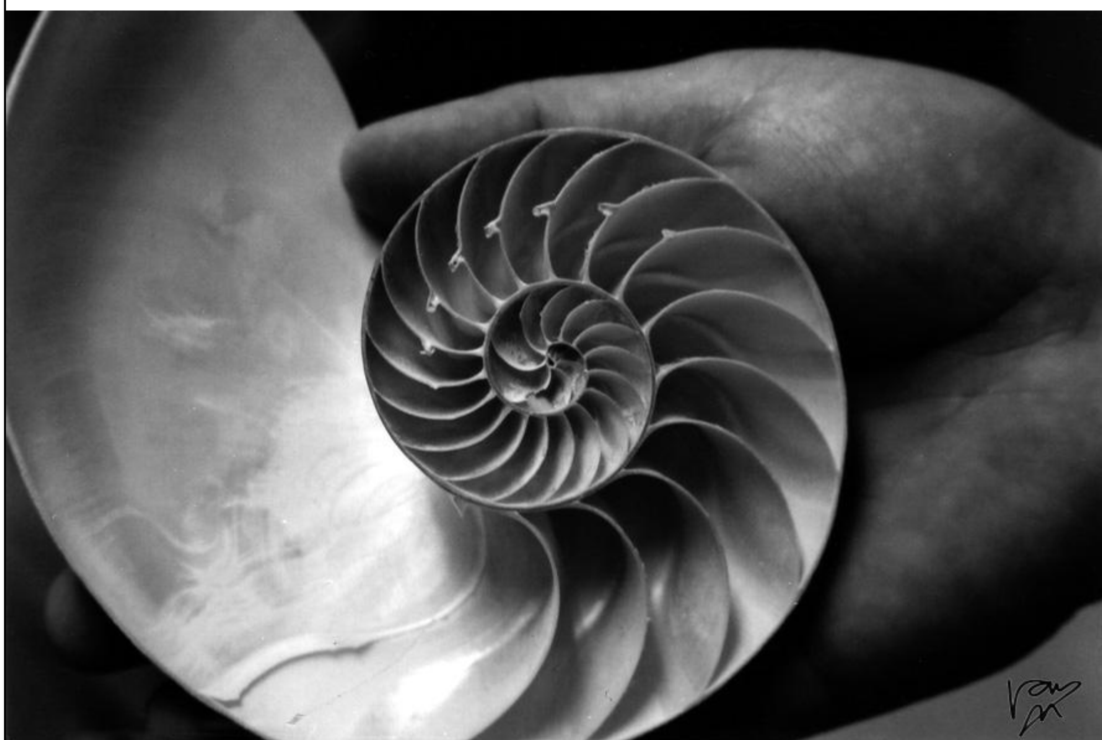
Lecture 23:
Integration: Cartesian and Polar



Learning Calculus With Geometry Expressions™

by L. Van Warren

Chapter 6: Integration Techniques

Lecture	Topic
23	Integration: Cartesian and Polar
24	Integration by Substitution
25	Integration by Parts
26	Area Between Curves

Calculus Inspiration

David Hilbert

PhD. Königsberg 1897
Göttingen University

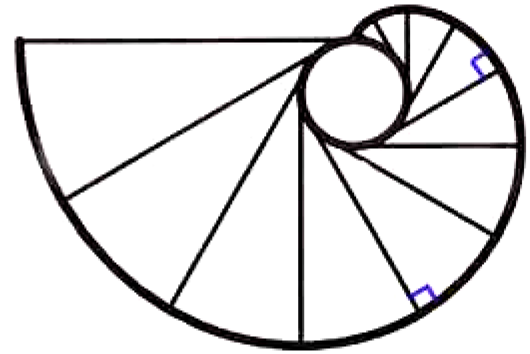
- Function Spaces
- Axiomization of Geometry
- The 23 Problems
- 76 Doctoral Students
  16296 Math Descendants

C. Felix Klein (57)

Julius Plücker

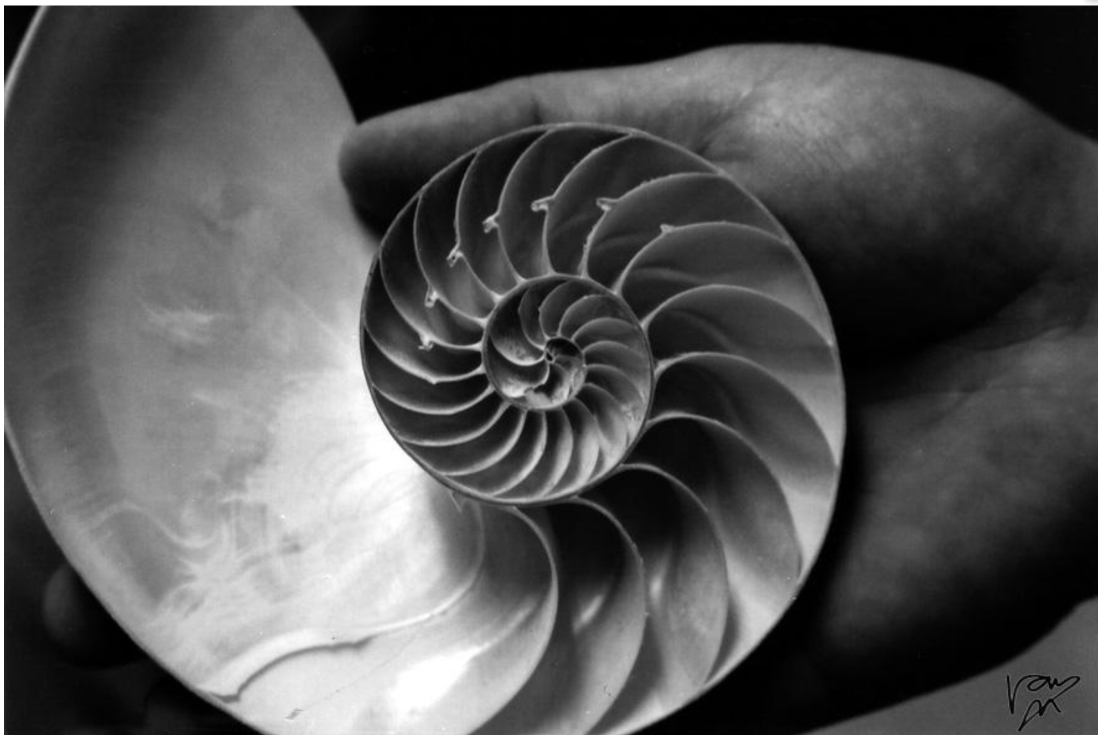
Christian Gerling

Carl F. Gauß (8)



Involute *of a circle*

Nautilus Shell



The Case of the Peculiar Limit:

Lecture23-CaseOfThePeculiarLimit.gx

In a previous lecture we discovered that the definite integral:

$$\int_0^a \frac{\sin(x)}{x}\,dx$$

Has no closed-form solution.

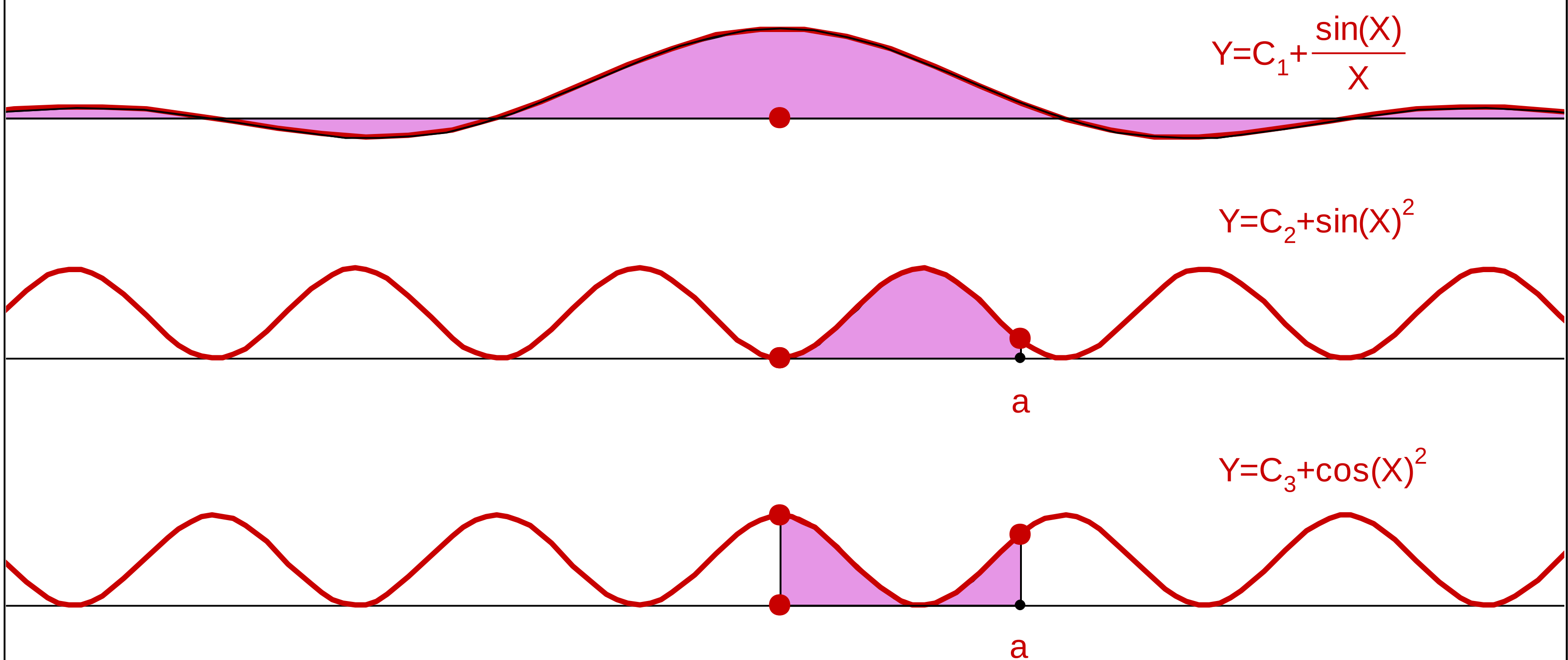
However we are pressed to find that value of the upper-limit a that most closely satisfies the following integral equation:

$$\int_{-\infty}^{+\infty} \frac{\sin(x)}{x}\,dx = \int_0^a \sin^2(x)\,dx + \int_0^a \cos^2(x)\,dx$$

The Case of the Peculiar Limit:

Lecture23-CaseOfThePeculiarLimit.gx

The integrals are displayed on the same page using constants C_1, C_2, and C_3. What value of a produces an area in the bottom two figures that is the same as the infinitely wide top area?



Exercise:

Guess the value of a.

The Case of the Peculiar Limit:

Lecture23-CaseOfThePeculiarLimit.wxm

Using Maxima™ we discover an interesting fact. Although the sin(x)/x integral has no finite solution it does have an infinite solution!

```
integrate(sin(x)/x,x,-inf,inf);
π
```

$$\int_{-\infty}^{+\infty} \frac{\sin(x)}{x}\,dx = \pi$$

We also note it is possible to combine the second two integrals:

$$\int_0^a \sin^2(x)\,dx + \int_0^a \cos^2(x)\,dx = \int_0^a \left[\sin^2(x) + \cos^2(x)\right] dx$$

$$= \int_0^a dx = a$$

These two facts simplify our original equation to: $\pi = a$

How close was this result to your original guess?

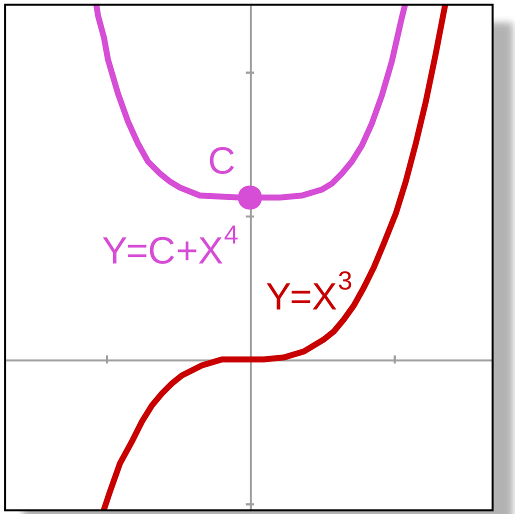
The Indefinite Integral:

Lecture23-IndefiniteIntegral.gx

The indefinite integral is a convenient form of antiderivative:

$$\int f(x)dx = F(x) \quad \text{where} \quad \frac{d}{dx} F(x) = f(x)$$

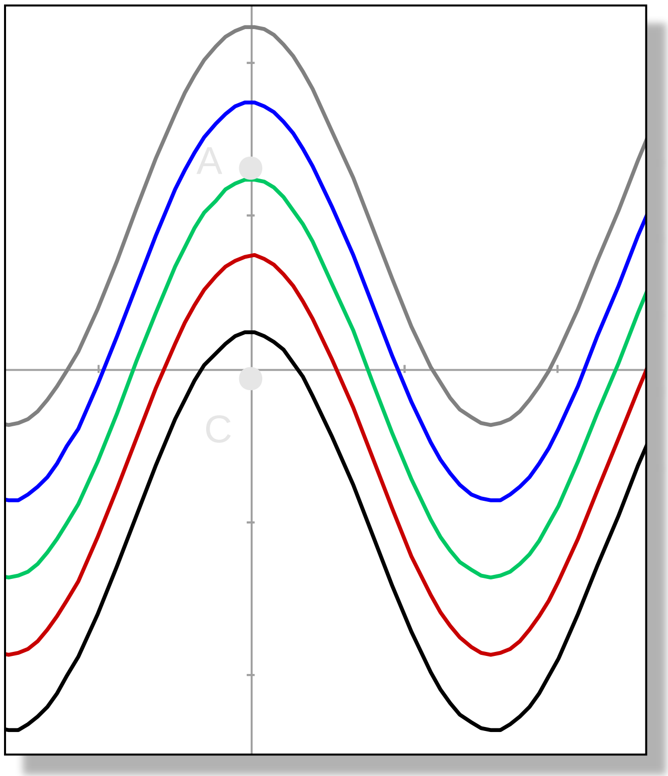
$$\int x^3 \, dx = \frac{x^4}{4} + C \quad \text{because} \quad \frac{d}{dx}\left(\frac{x^4}{4} + C\right) = x^3$$



Exercises:
1) Drag C.
2) Redo with $f(x) = e^x$

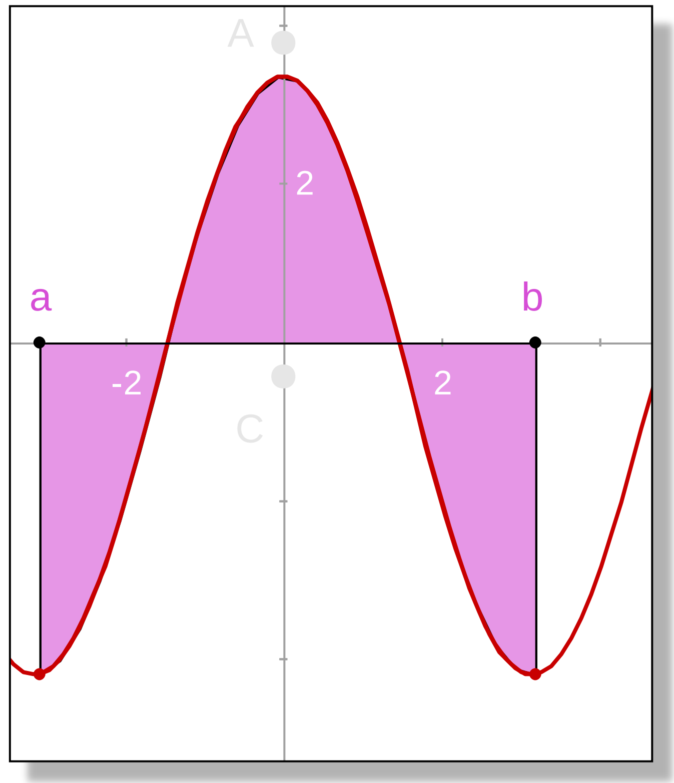
Indefinite Vs. Definite Integral:

A Family of Curves:



VS.

A Specific Area:



$$\int f(x)dx = F(x)$$

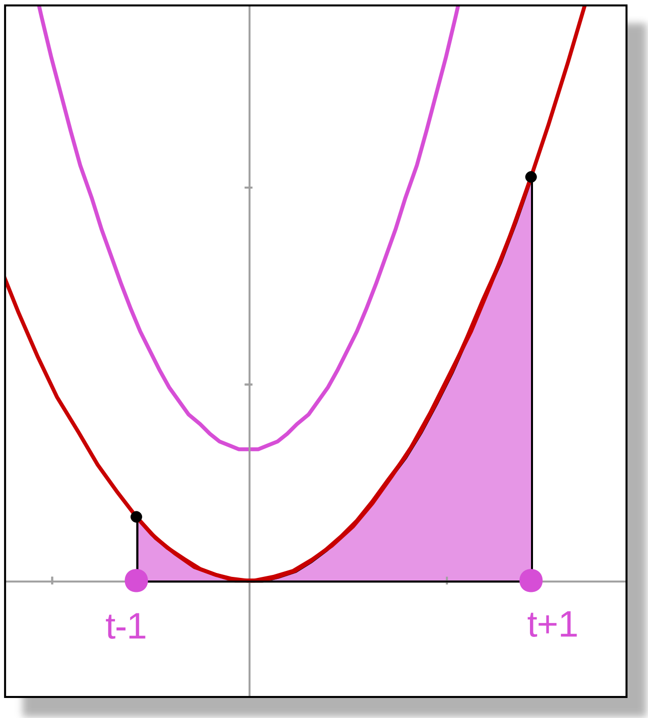
Lecture23-FamilyOfCurves.gx

$$\int_a^b f(x)dx = F(b) - F(a)$$

Lecture23-ASpecificArea.gx

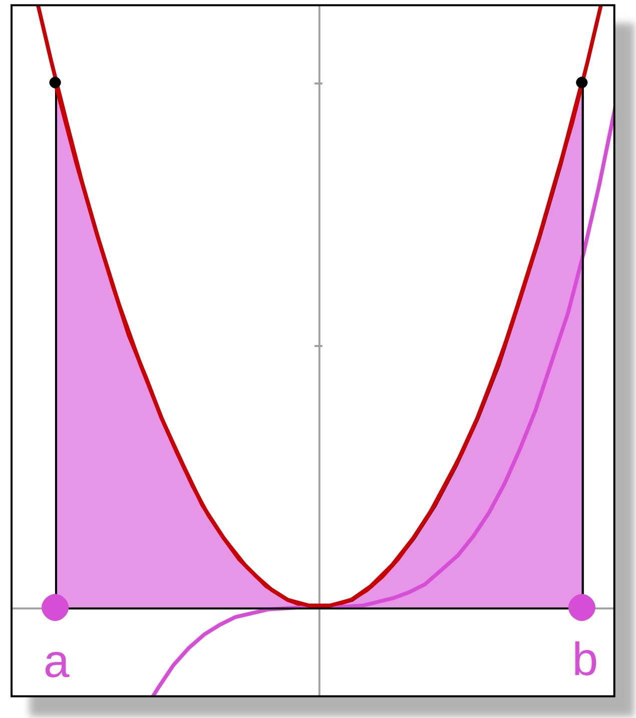
Variable vs. Constant Limits:

Variable Limits



vs.

Constant Limits



$$\int_{t-1}^{t+1} x^2\,dx = 2t^2 + \frac{2}{3}$$

Lecture23-VariableLimits.gx

$$\int_{-2}^{2} x^2\,dx = \left.\frac{x^3}{3}\right|_{-2}^{2} = \frac{16}{3}$$

Lecture23-ConstantLimits.gx

Indefinite Integral – No Guarantee of Existence:

Given f(x) continuous on [a..b] we can always find f'(x):

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

But

$$\int f(x)dx = \ ?$$

Finding the derivative is deterministic,

BUT

There is no guarantee that a solution even exists.
Finding the antiderivative, the indefinite integral is a SEARCH.

Exercise: Comment on how this might relate to invertibility.

Search Strategy:

One strategy for finding integrals is to differentiate functions and go backwards.

That is, the function you are differentiating is some other problem's integral!

One can generate vast tables this way.

$\frac{d}{dx}\sin(x) = \cos(x)$	$\int \cos(x)dx = \sin(x)+C$
$\frac{d}{dx}\cos(x) = -\sin(x)$	$\int \sin(x)dx = -\cos(x)+C$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\int \sec^2(x)dx = \tan(x)+C$
$\frac{d}{dx}\cot(x) = -\csc^2(x)$	$\int \csc^2(x)dx = \cot(x)+C$
$\frac{d}{dx}x^n = nx^{n-1}$	$\int nx^{n-1}\,dx = x^n+C$
$\frac{d}{dx}\frac{x^{n+1}}{n+1} = x^n$	$\int x^n\,dx = \frac{x^{n+1}}{n+1}+C$
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	$\int x^{-1}\,dx = \ln(x)+C$

Search Strategy:

Let's say we wanted to find the integral of a function that has never been integrated in closed-form, like sin(x)/x. We might start by differentiating functions that give results that look like sin(x)/x:

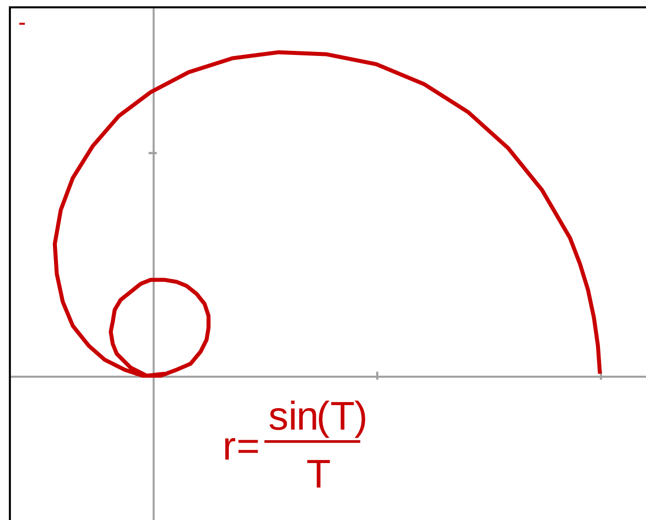
$$\int \frac{\sin(x)}{x}dx = ?(x)+C \rightarrow \frac{d}{dx}?(x) = \frac{\sin(x)}{x}$$

```
diff(-cos(x),x);
sin(x)

diff(log(x),x);
1/x

diff(-cos(log(x)),x);
sin(log(x))/x
```

Lecture23-SinXOverX-A.wxm



Lecture23-Nautilus.gx

List Comprehensions:

Lecture23-ListComprehensions.wxm

A list comprehension is just an expression that creates a list:

Here are some examples:

```
create_list(2*i,i,1,10);
```

$[2,4,6,8,10,12,14,16,18,20]$

```
h[i]:=x^i;
```

$h_i := x^i$

```
create_list(h[i],i,[1,2,10]);
create_list(h[i],i, 1,10    );
create_list(h[i],i,create_list(fib(j),j,1,10));
```

$[x, x^2, x^{10}]$

$[x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}]$

$[x, x, x^2, x^3, x^5, x^8, x^{13}, x^{21}, x^{34}, x^{55}]$

```
create_list(integrate(sin(x)/h[i],x),i,1,5);
```

$\left[\int \frac{\sin(x)}{x}dx, \int \frac{\sin(x)}{x^2}dx, \int \frac{\sin(x)}{x^3}dx, \int \frac{\sin(x)}{x^4}dx, \int \frac{\sin(x)}{x^5}dx\right]$

Matrices of Functions:

Lecture23-MatricesOfFunctions.wxm

We can also generate a matrix of functions, integrals and limits.

```
h[i,j]:=j*sin(i*x);
```

$$h_{i,j} := j\, \sin(i\, x)$$

```
H: genmatrix(h,3,3);
```

$$\begin{bmatrix} \sin(x) & 2\sin(x) & 3\sin(x) \\ \sin(2\,x) & 2\sin(2\,x) & 3\sin(2\,x) \\ \sin(3\,x) & 2\sin(3\,x) & 3\sin(3\,x) \end{bmatrix}$$

```
HI:integrate(H,x);
```

$$\begin{bmatrix} -\cos(x) & -2\cos(x) & -3\cos(x) \\ -\frac{\cos(2\,x)}{2} & -\cos(2\,x) & -\frac{3\cos(2\,x)}{2} \\ -\frac{\cos(3\,x)}{3} & -\frac{2\cos(3\,x)}{3} & -\cos(3\,x) \end{bmatrix}$$

Exercises:

1) Differentiate [H] with respect to x.
2) Take the limit of [H] as x→a.
3) Create a matrix of functions [G] by replacing sin(x) in h[i,j] with cos(x).
4) Create a 5 x 5 version of the [H] and [G].

Numerical Integration:

Lecture23-NumericalIntegration.gx

For cases where a closed-form solution does not exist, a numerical approach, based on the Riemann Sum is useful:

$$F(x) = \int_a^b cos(x)\,dx \cong \sum_{i=1}^{n} cos(x_i^*)\cdot h$$

where

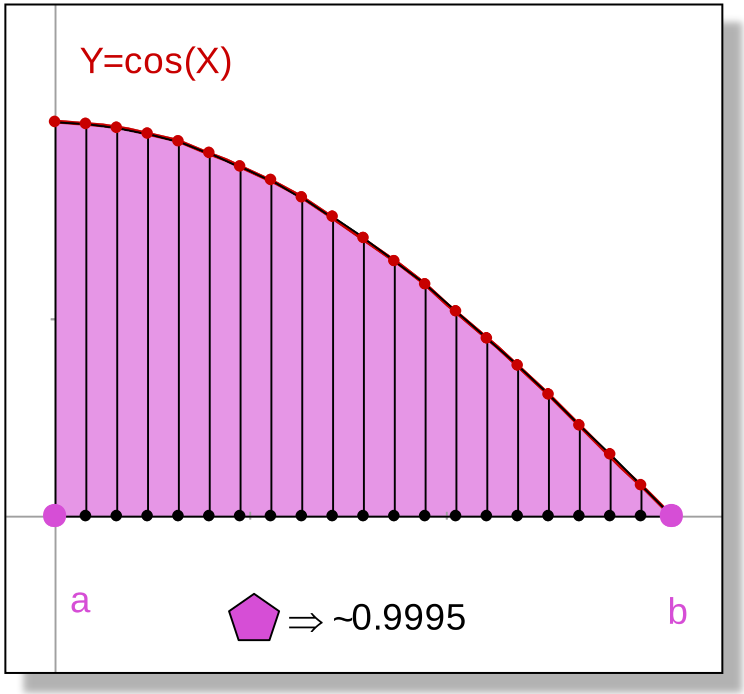
$$h = \frac{(b-a)}{n}$$

and

$$x_{i+1} = x_i + h$$

and

$$x_i^* = \frac{x_{i-1} + x_i}{2}$$



Numerical integration is also called "*quadrature*". Can you speculate why?

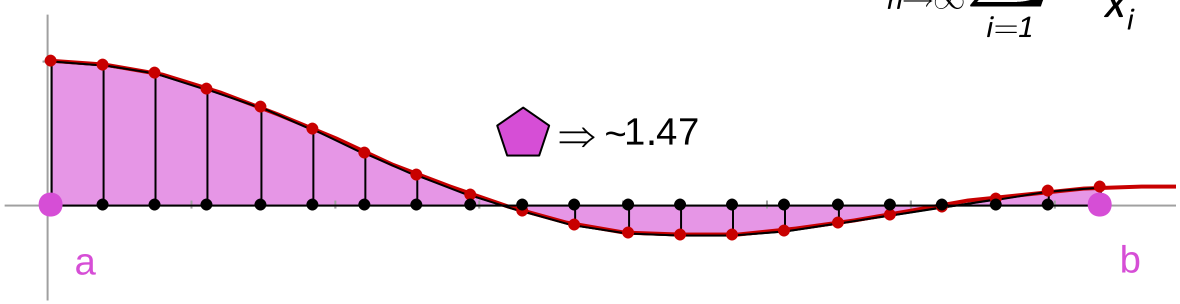
Numerical Solutions:

Lecture23-Quadrature.gx

Numerical solutions require a finite number of terms for the computation to *halt*. If the computation doesn't halt, the problem is *undecideable* and no solution is possible using that approach.

Showing that a numerical approximation converges to an exact solution is vital for correctness. This issue is explored further in numerical analysis, and uses *limit* techniques learned here.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sin(x_i^*)}{x_i^*}\cdot h$$

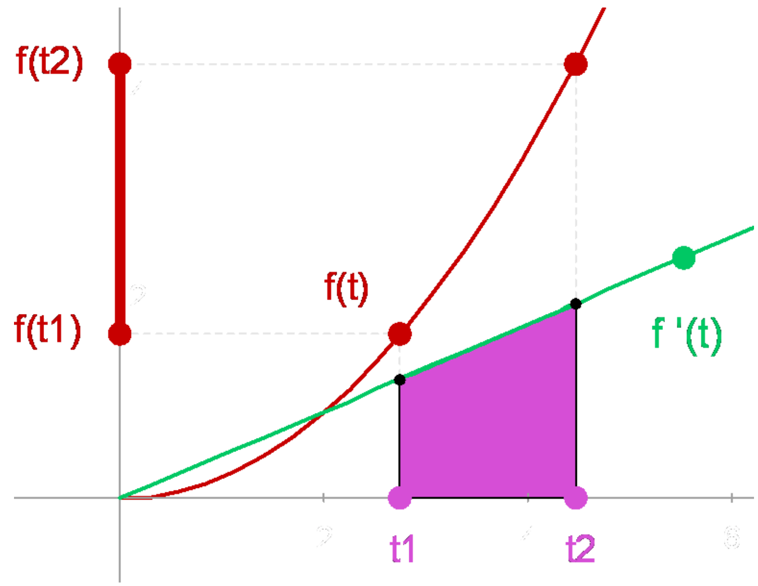


Net Change Theorem

Lecture23-NetChangeTheorem.gx

From the Fundamental Thereom of Calculus we observe, the integral of the derivative of f(t) in an interval $[t_1..t_2]$ is the net change of f(t) in that interval:

$$\int_{t_1}^{t_2} f'(t)dt = \int_{t_1}^{t_2} \frac{df(t)}{dt} dt = \int_{t_1}^{t_2} df(t) = f(t_2) - f(t_1)$$



Net Change Car Crash

Lecture23-NetChangeLinear.gx

Consider a car whose velocity v(t) over time is given by the curve:

$$v(t) = \frac{ds(t)}{dt} = \frac{60[mph]}{6.4[s]} \cdot t[s]$$

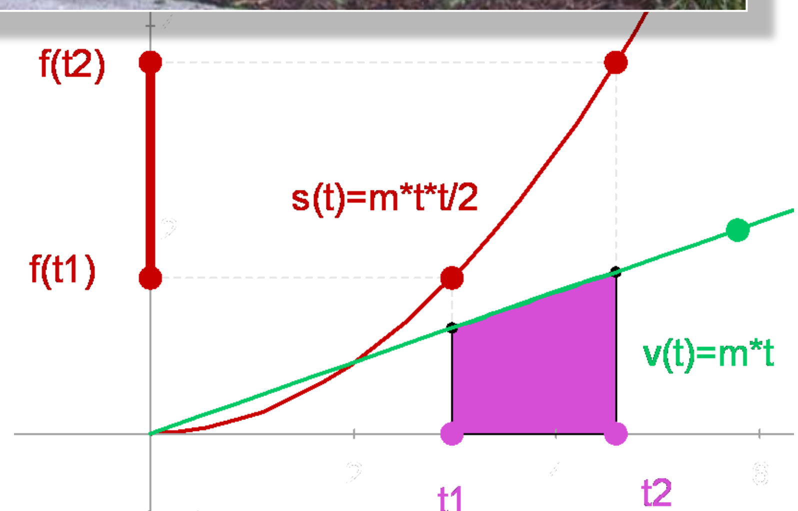
WATCH UNITS!

$$= \frac{88[fps]}{6.4[s]} \cdot t[s] = 13.75 \cdot t$$

How far does it travel as it accelerates from 0 to 60 mph?

$$\int_0^{6.4} v(t)dt = 13.75 \int_0^{6.4} t\,dt =$$

$$\frac{13.75}{2} \cdot t^2 \Big|_0^{6.4} \cong 282[ft]$$



Exercises:

1) Adjust t_1, t_2 and m to discover the distance traveled.
2) Would a car travel this far while decelerating? Theorize.

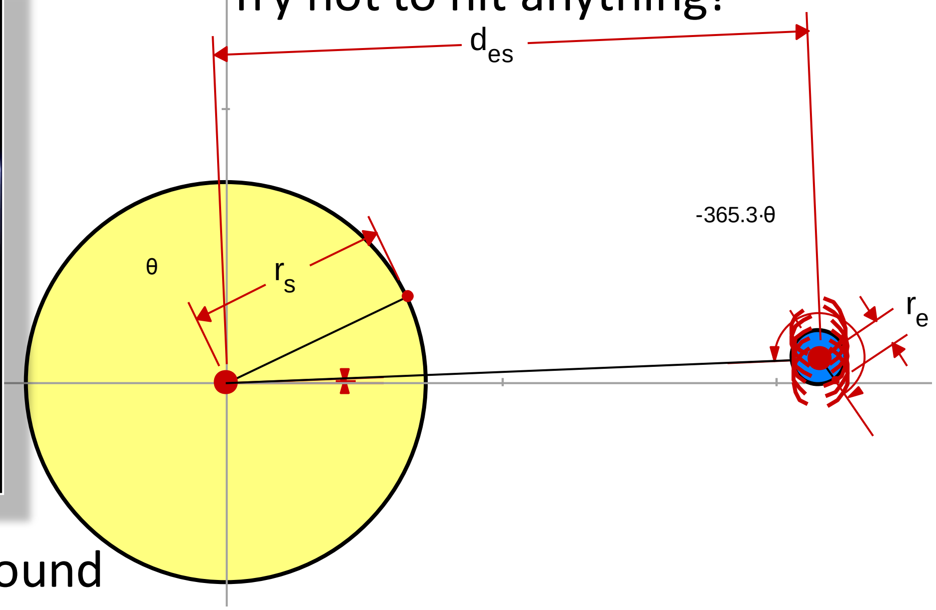
Angular Velocity

Lecture23-SunEarthGeometry.gx

Right now you are traveling ~ 1,000 mph around the center of the earth and ~ 67,000 mph around the center of the sun.

Try not to hit anything!

- Don Davis, NASA



The earth rotates 365.25 times around its own axis for every revolution around the sun.

Exercises:

1) Click the play button to animate θ, the angle of the earth around the sun.
2) Adjust r_s, r_e and d_{es} to realistic values where 1 unit = 1 million miles.
3) What points on a circular earth appear to intersect themselves? How often?
4) Why would this be important for meteor impact, or for time travel?

Integration in Radial Coordinates

Lecture23-AreaUnderPolarCurve.gx

When we integrate in Cartesian coordinates we drew rectangles and trapezoids. When we integrate in polar coordinates we draw triangles. Consider the following equation of a "line" in polar coordinates:

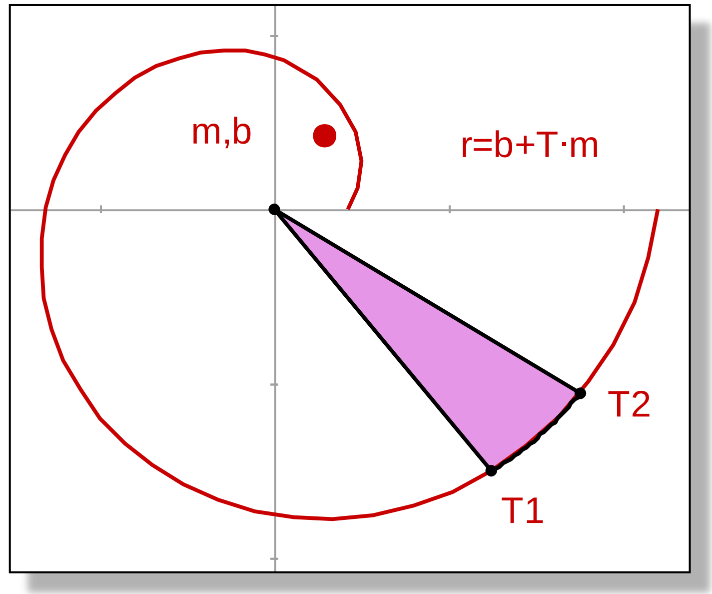
$$r(T) = m\cdot T + b$$

The height of a given triangle: $r(T_i)$

The base of a given triangle: $\Delta s = r(T_i)\,\Delta T$

The area of the i_{th} triangle is:

$$A_i = \frac{1}{2} b_i h_i = \frac{1}{2} \Delta s \cdot r_i = \frac{1}{2} r_i \Delta T \cdot r_i = \frac{1}{2} r_i^2 \Delta T$$



Taking the limit we have the area is:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} r(T_i)^2 \cdot \Delta T = \frac{1}{2} \int_{T_1}^{T_2} r(T)^2 \, dT$$

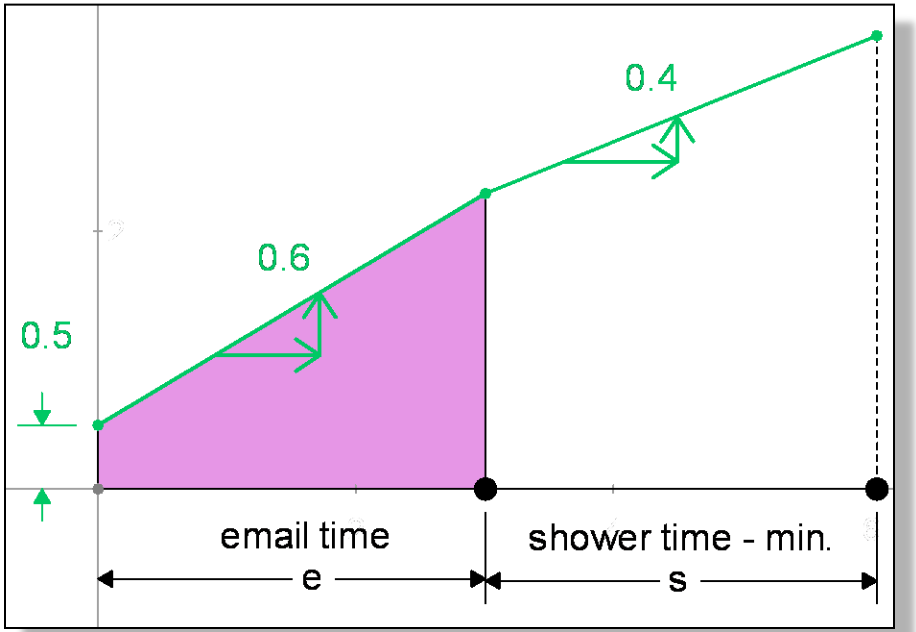
Exercises:

1) Drag the points (m,b), T1 and T2 to form the area of a quarter circle.
2) Compute the real area.

Everything But The Kitchen Sink

Lecture23-KitchenSink.gx

While attempting to cook, a certain professor has left the kitchen faucet on "just for a second". He checks his email for three minutes and starts a shower. There is half a gallon of water in the sink when he starts to check his email. The capacity of the sink is 4.85 gallons. Water flows into the sink at 0.6 gallons per minute while he checks his email, but slows to 0.4 gallons per minute while he showers.



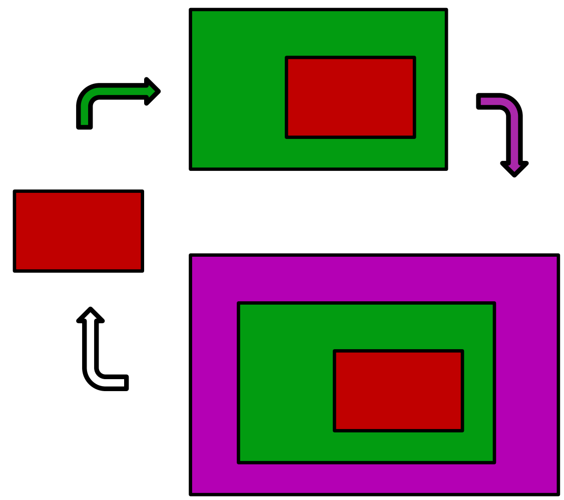
Exercises:

1) Set up the symbolic integral.
2) Compute the overflow time.
3) Complete the diagram.
4) How long can the shower be?

$$\frac{d}{dx}f(x)$$

$$f(x)$$

$$\int \frac{d}{dx}f(x)\,dx$$



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