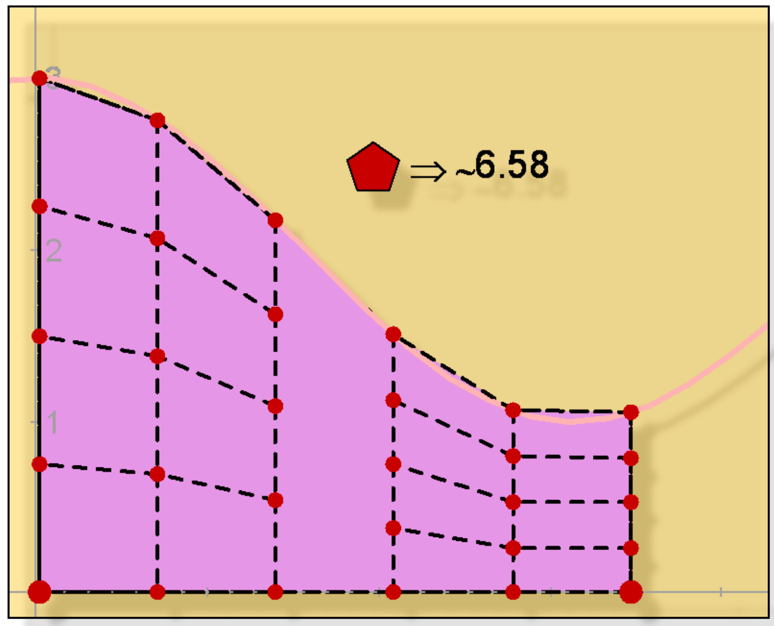
Learning Calculus With Geometry Expressions™

by L. Van Warren

Lecture 20: Integration: Area and Distance



Chapter 5: Integration

LECTURE	TOPIC
19	ANTIDERIVATIVES
20	**INTEGRATION: AREA AND DISTANCE**
21	THE DEFINITE INTEGRAL
22	FUNDAMENTAL THEOREM OF CALCULUS

Inspiration

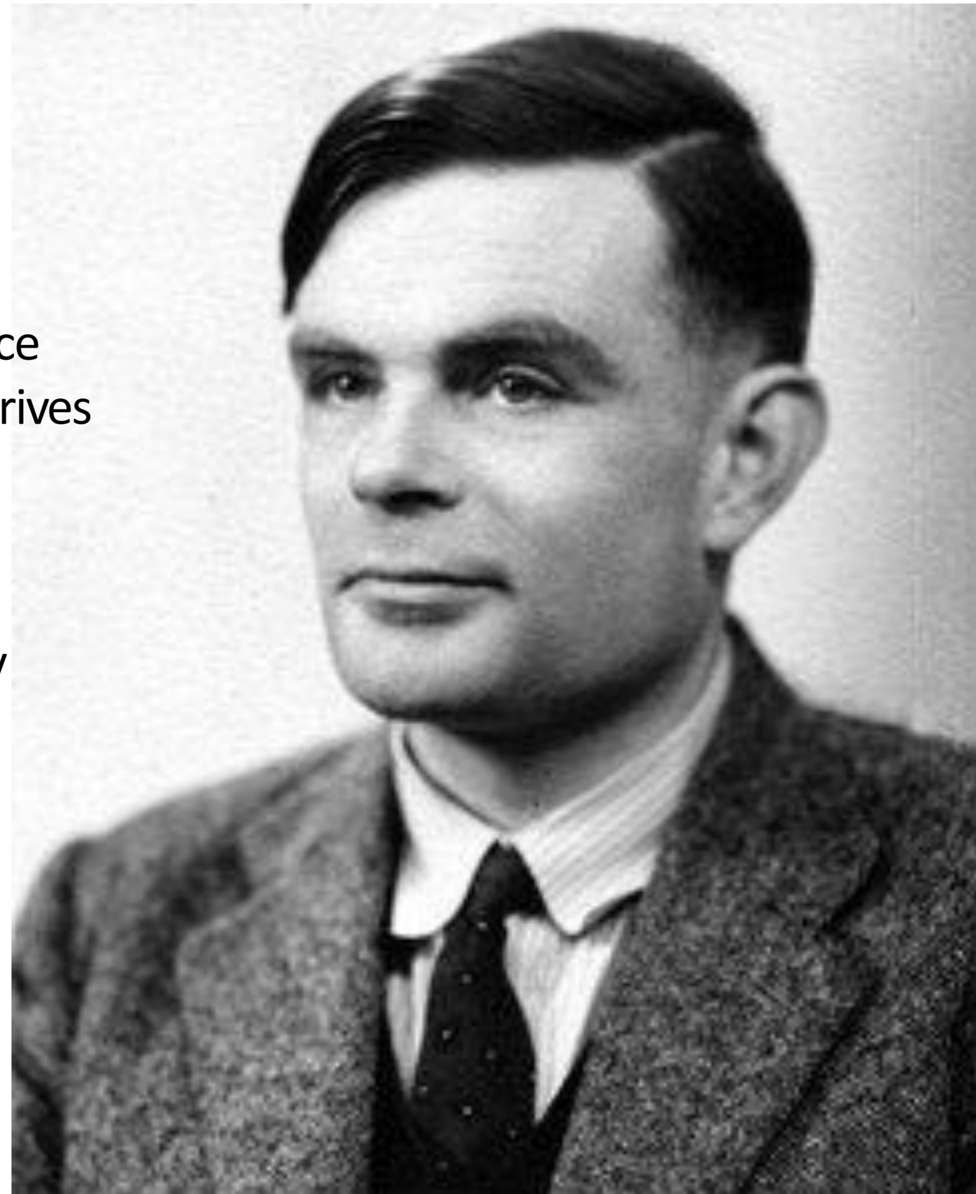
Alan Turing   1912 - 1954
Cambridge / Princeton

- Father of Modern Computer Science
  From Which Computer Algebra Derives

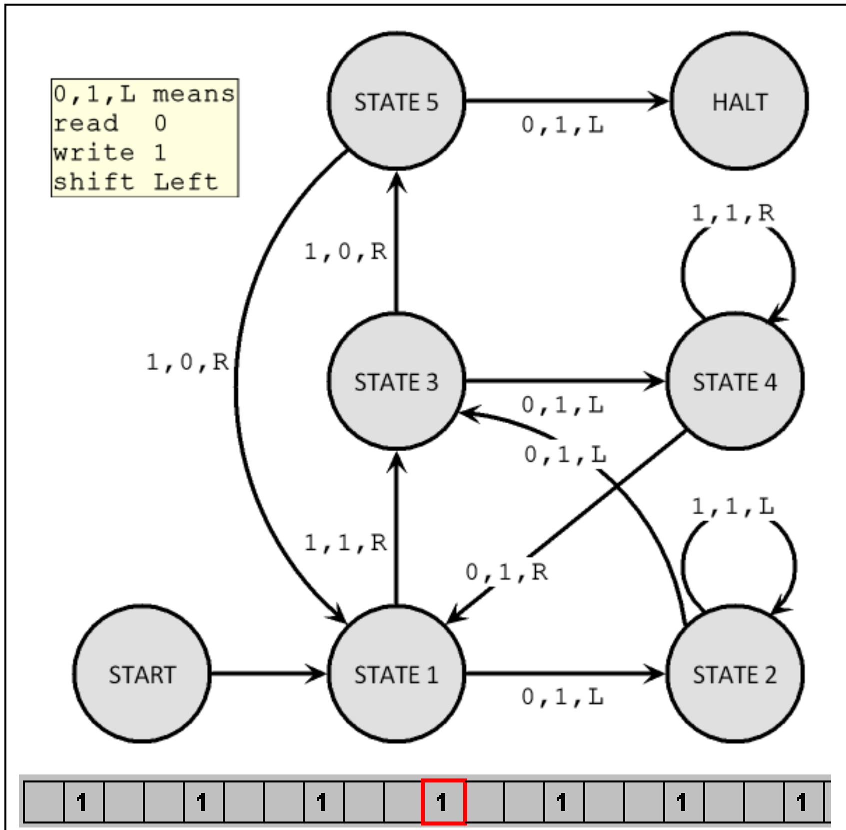
- Created Turing Machine

- Devised Turing Test : Computability

- Translated Gödel's Work
  Into Equivalent Machines



Turing Machines:



-This five-state "Busy Beaver" halts in only 47,176,870 Steps!

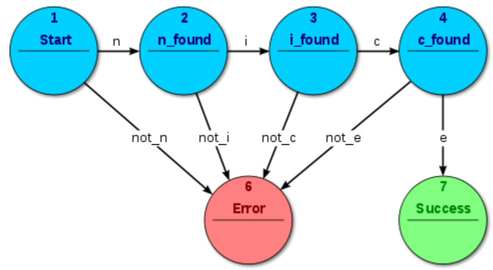
A Turing Machine (TM) consists of a TAPE and a PROGRAM executing in discrete STEPS.

The TAPE consist of sequential cells, each containing symbols from a finite alphabet. The PROGRAM is a graph of states.

At each STEP, a TM may:

a) Read & Write the current symbol.
b) Change state .
c) Move the TAPE left or right.

Proofs as “Devices Under Test”:



Our responsibility in mathematics include:

1) Uncover truth by proving statements as true or false OR
2) Prove that 1) cannot be proven for a given statement`.

- Absolute proof is subject to the limitations of Gödel.
- Practical proof verifies consistency of a statement within the assumptions that are themselves proven practically.
- A Turing machine that halts constitutes a proof of a given finite state machine and a given input.

Proofs as “Devices Under Test”:

The proof of a mathematical statement is analgous to the concept of “Device Under Test” in electrical circuits.

- A statement can be verified to a given certainty statistically.
- This leads to the possibility of abstracting a proof in terms of input and output relationships.
- A function that produces a certain output for a given input is verified for that input.
- If a function is verified over 95% of its input space, then we have a 95% certainty that the function performs as expected, that is, is true to our expectation.

Stalking the Wild Asparagus:

- *Nine percent of respondents report that Asparagus reminds them of the integral sign.*

In the last lecture we examined antiderivatives – shown as equivalent to indefinite integrals.

This lecture is about finding the area under a curve. We will do this by using the definite integral, we will start by reviewing some important results.

$$\int_1^2 x\,dx = \left.\frac{x^2}{2}\right|_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = 2 - \frac{1}{2} = \frac{3}{2}$$

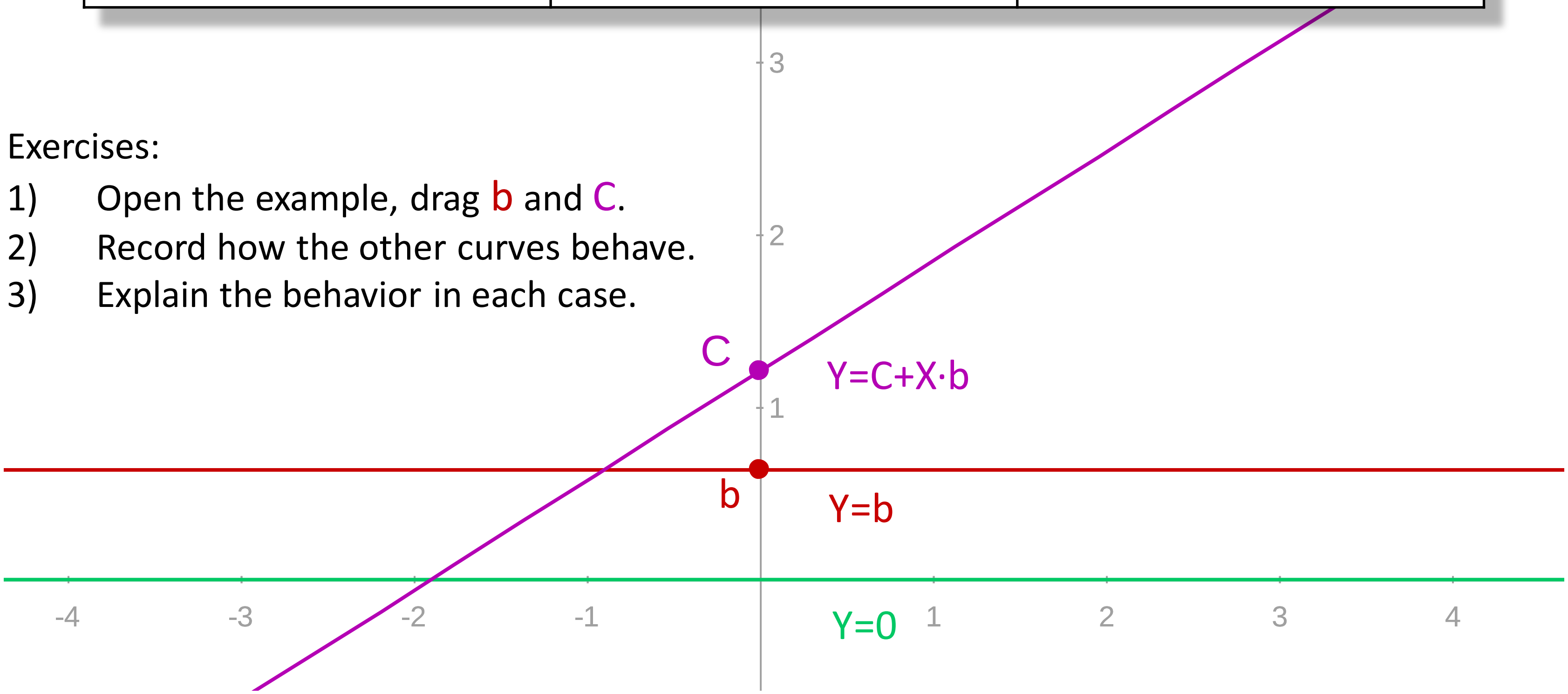
Lecture20-IntegralOfConstant.gx

Derivatives & Integrals: Constant Case

Derivative	Function	Integral
$\frac{d}{dx}b = 0$	b	$\int b\,dx = bx + C$

Exercises:

1) Open the example, drag b and C.
2) Record how the other curves behave.
3) Explain the behavior in each case.



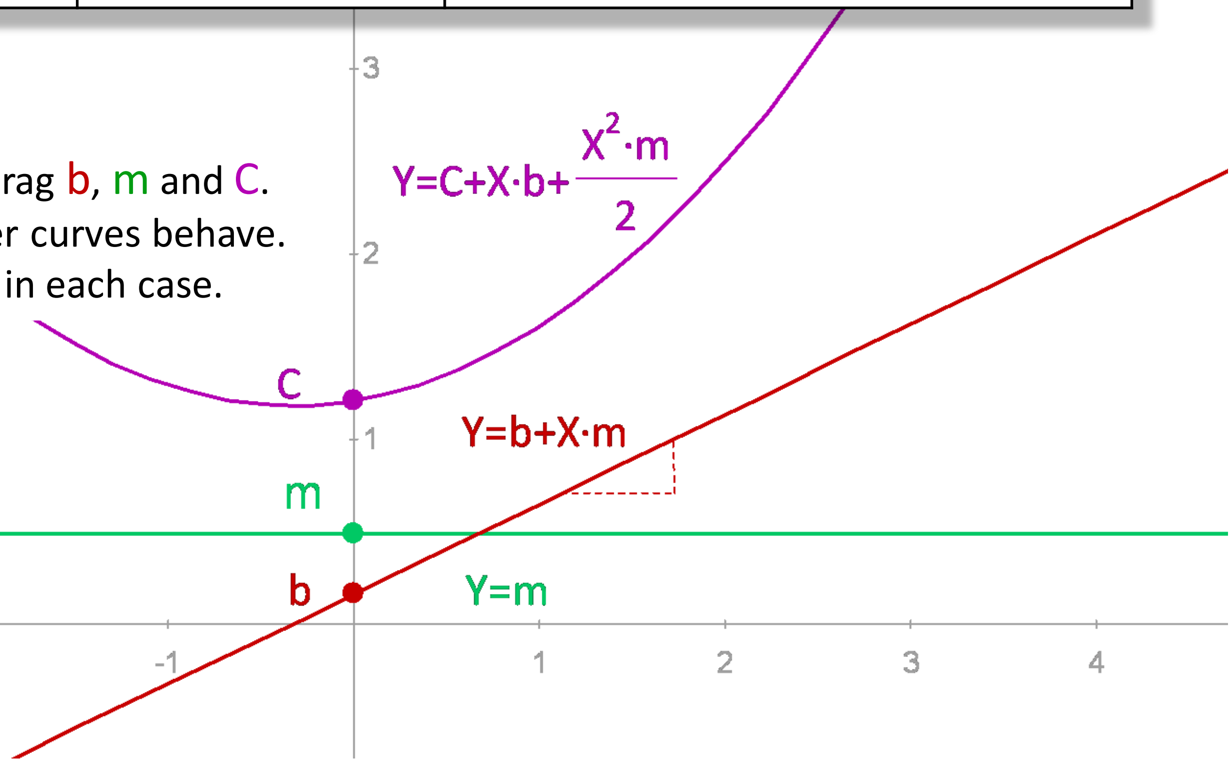
Lecture20-IntegralOfLinear.gx

Derivatives & Integrals: Linear Case

Derivative	Function	Integral
$\frac{d}{dx}(mx+b)=m$	$mx+b$	$\int(mx+b)dx=\frac{m}{2}x^2+bx+C$

Exercises:

1) Open the example, drag b, m and C.
2) Record how the other curves behave.
3) Explain the behavior in each case.



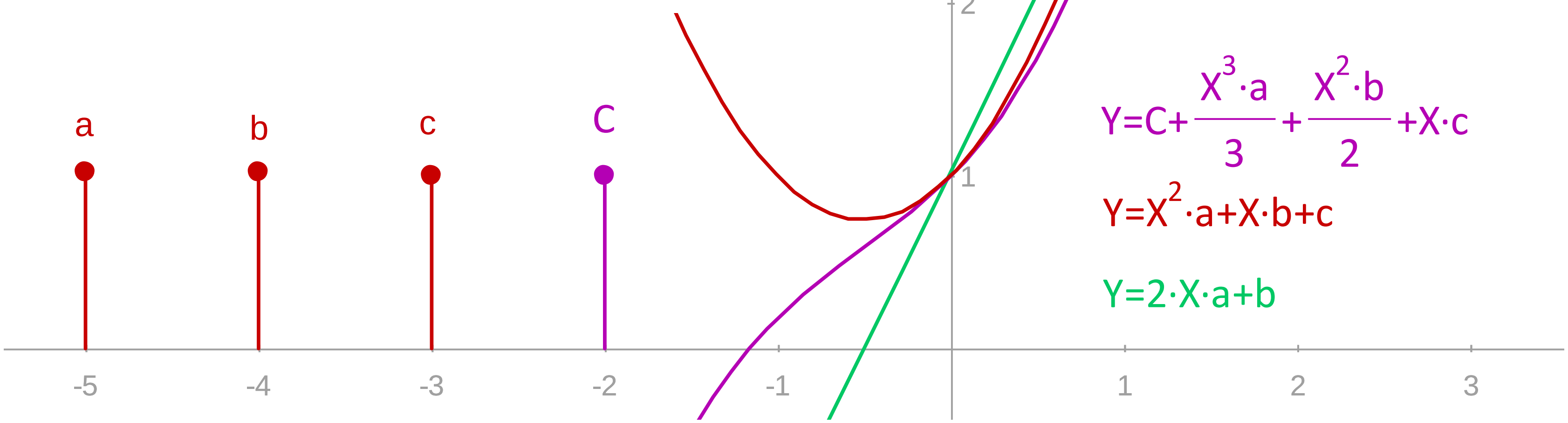
Lecture20-IntegralOfQuadratic.gx

Derivatives & Integrals: Quadratic Case

Derivative	Function	Integral
$2ax+b$	ax^2+bx+c	$\frac{a}{3}x^3+\frac{b}{2}x^2+cx+C$

Exercises:

1) Drag a, b, c, and C.
2) Record how the other curves behave.
3) Explain the behavior in each case.



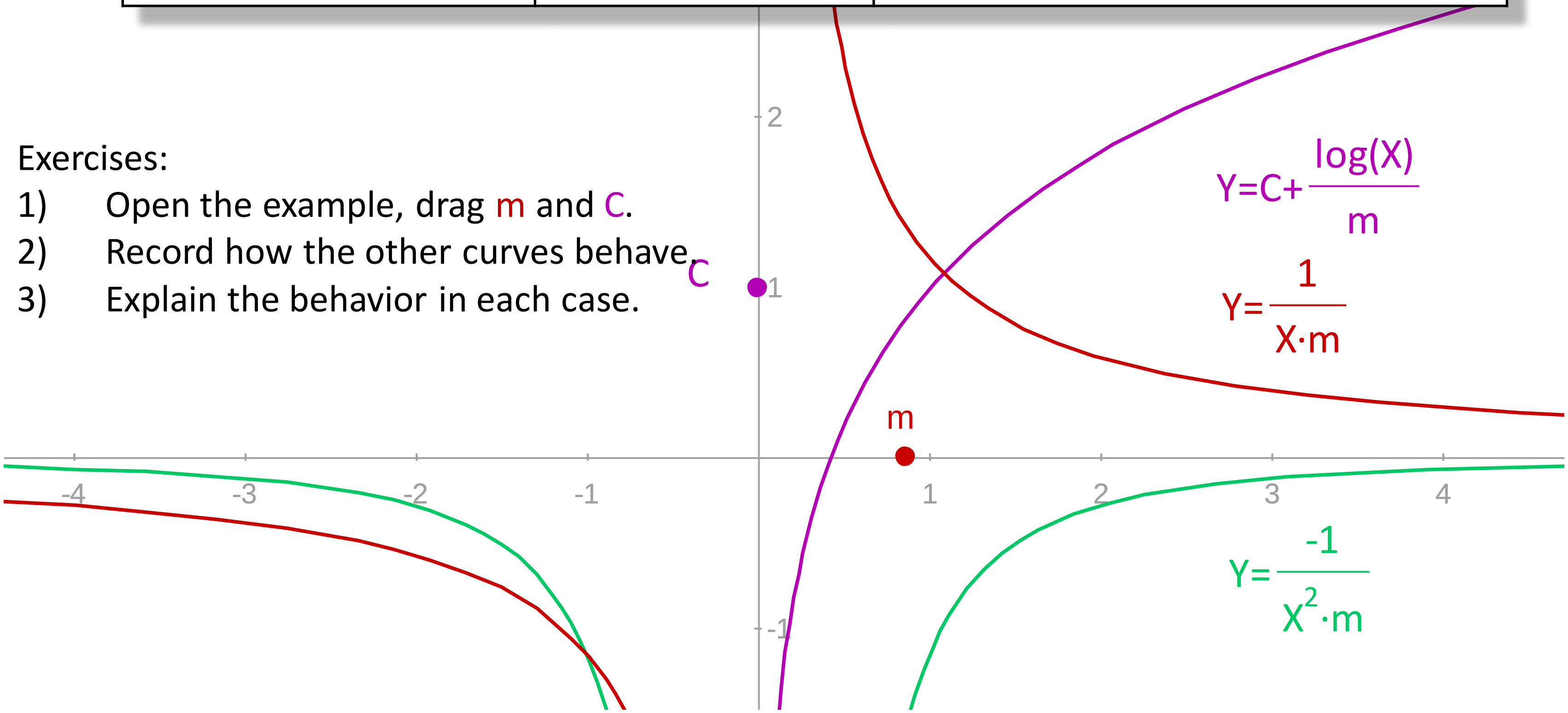
Lecture20-IntegralOfInverse.gx

Derivatives & Integrals: Inverse Case

Derivative	Function	Integral
$\frac{d}{dx}\left(\frac{1}{mx}\right)=\frac{-1}{mx^2}$	$\frac{1}{mx}$	$\int \frac{1}{mx}dx=\frac{1}{m}log(x)+C$

Exercises:

1) Open the example, drag m and C.
2) Record how the other curves behave.
3) Explain the behavior in each case.



Derivatives & Integrals: Other Cases

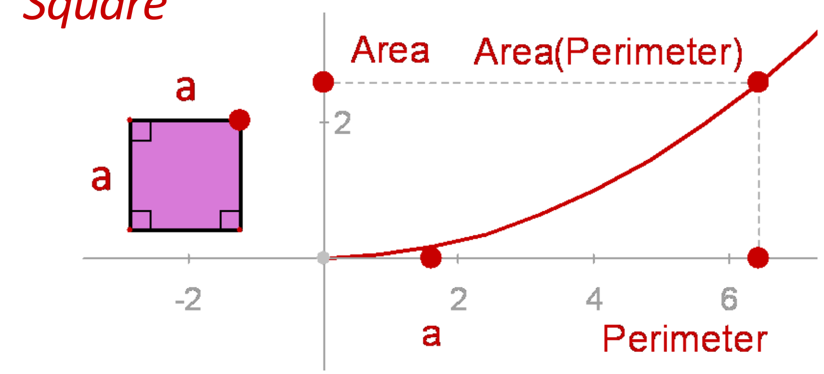
Derivative	Function	Integral
nax^{n-1}	ax^n	$\frac{a}{n+1}x^{n+1}+C$
$A\cos(x)$	$A\sin(x)$	$-A\cos(x)+C$
$A\omega\cos(\omega t+\phi)$	$A\sin(\omega t+\phi)$	$-\frac{A}{\omega}\cos(\omega t+\phi)+C$
e^x	e^x	e^x
$\frac{1}{x}$	$\ln(x)$	$x\ln(x)-x$

Exercise: Create a Geometry Expressions™ example for each case above.

Area and Perimeter of Common Figures

<table>
<tr><th>Figure</th><th>Perimeter</th><th>Area</th></tr>
<tr><td>

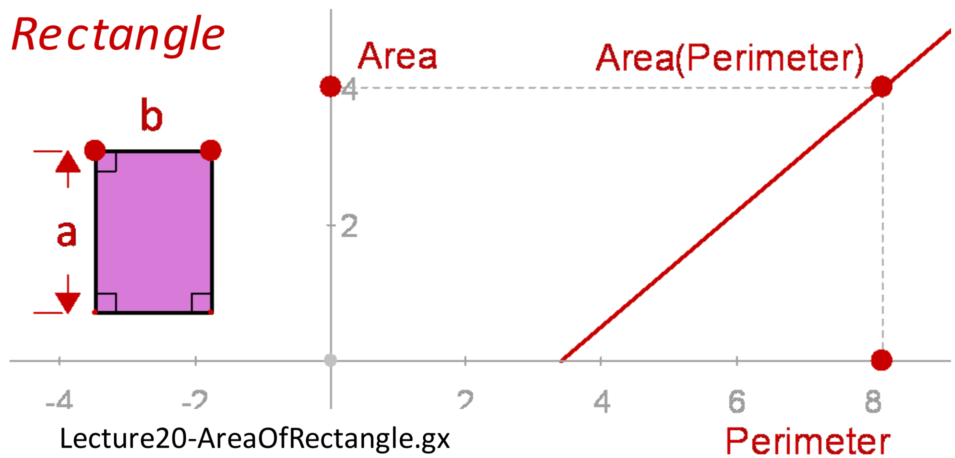
Square



Lecture20-AreaOfSquare.gx

</td><td>$4a$</td><td>a^2</td></tr>
<tr><td>

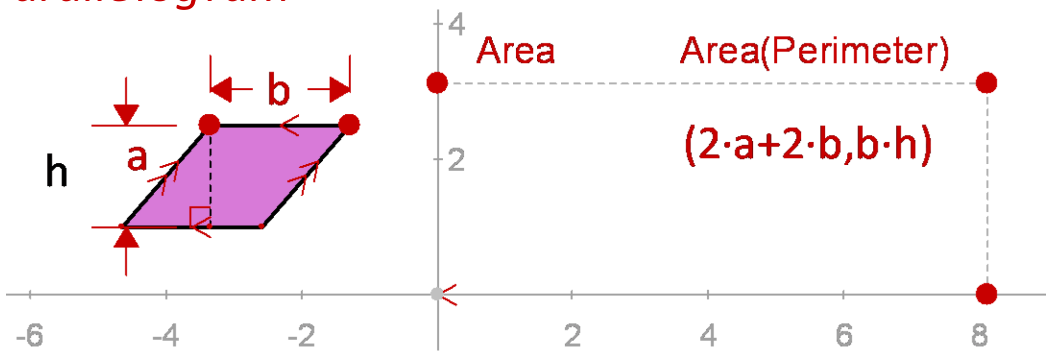
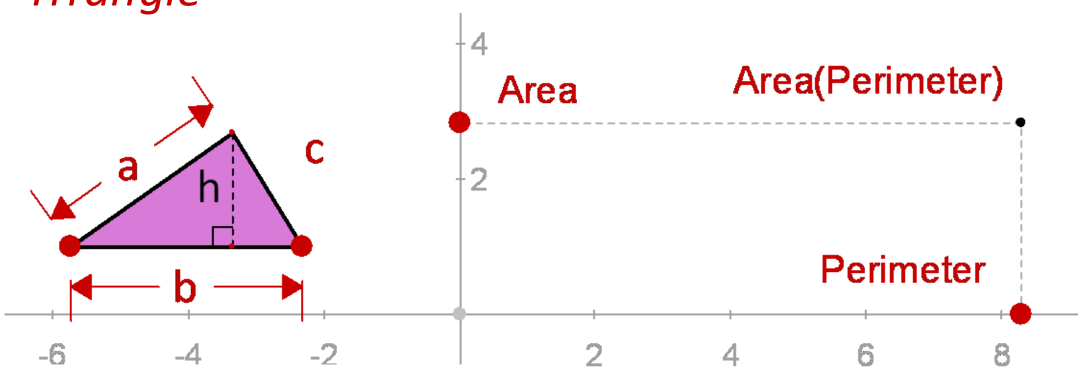
Rectangle



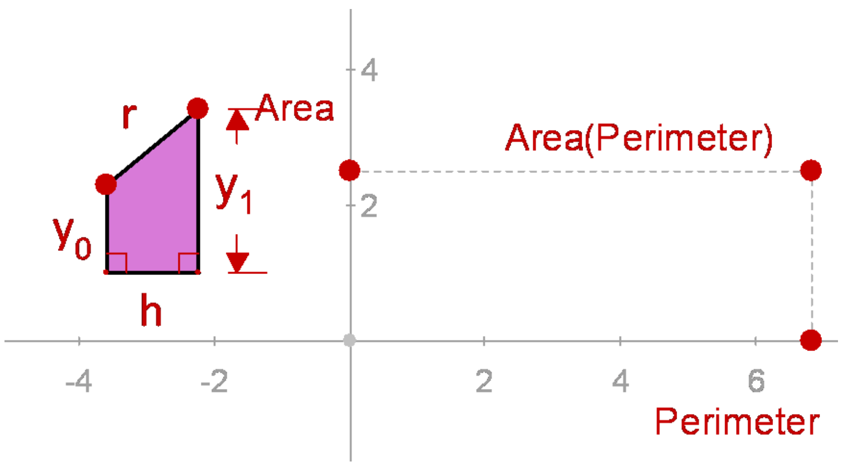
Lecture20-AreaOfRectangle.gx

</td><td>$2a+2b$</td><td>ab</td></tr>
</table>

Area and Perimeter of Common Figures

Figure	Perimeter	Area
Parallelogram  Lecture20-AreaOfParallelogram.gx	$2a+2b$	bh
Triangle  Lecture20-AreaOfTriangle.gx	$a+b+c$	$\frac{1}{2}bh$

Area and Perimeter of Common Figures

Figure	Perimeter	Area
Right Trapezoid  Lecture20-AreaOfTrapezoid.gx	$y_0 + h +$ $y_1 + r$ $r =$ $\sqrt{h^2 + (y_1 - y_0)^2}$	$h\dfrac{(y_0 + y_1)}{2}$

Lecture20-AreaUnderCurve.gx

Area Under a Curve

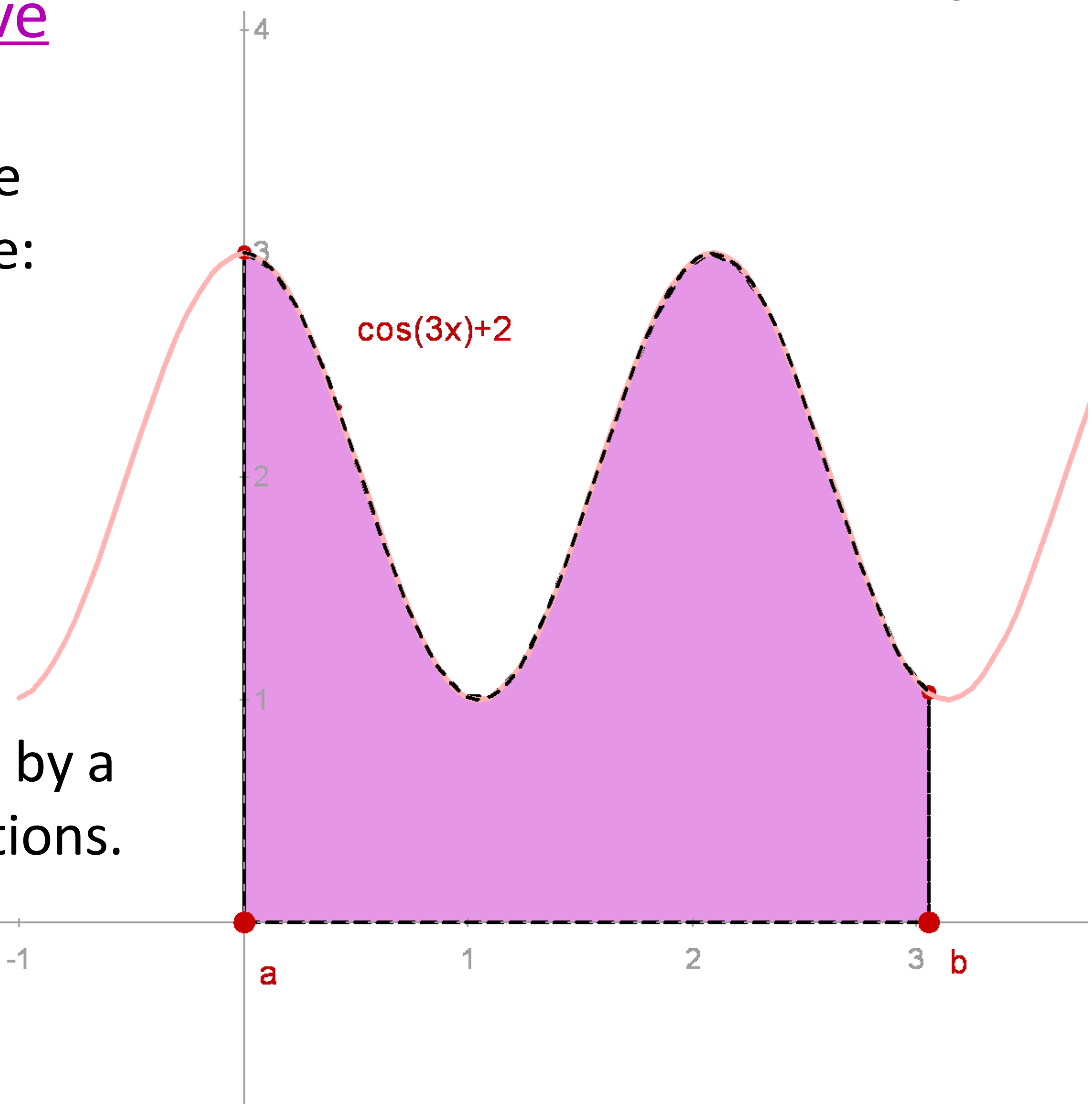
We desire to find the area under the curve:

$$y = cos(3x) + 2$$

This is written:

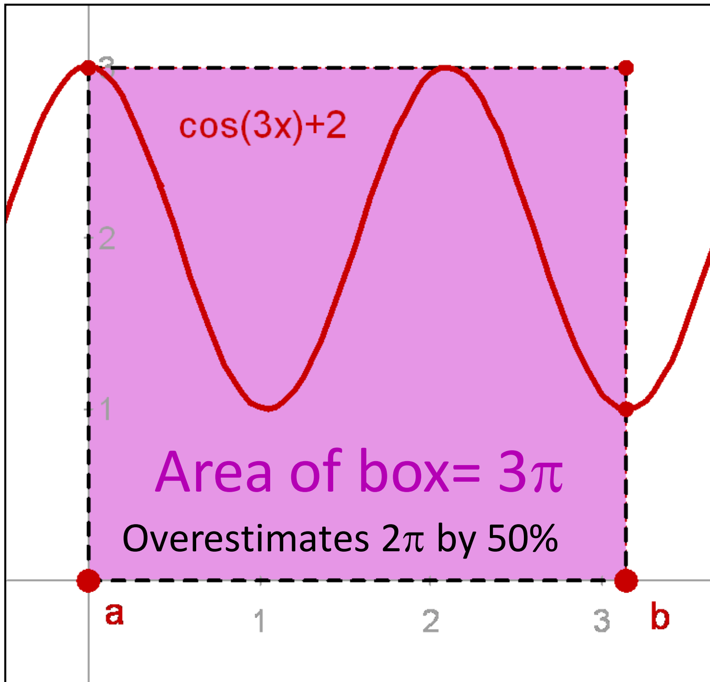
$$\int_{x=a}^{x=b} cos\ \ 3x\ \ +2\ \ dx$$

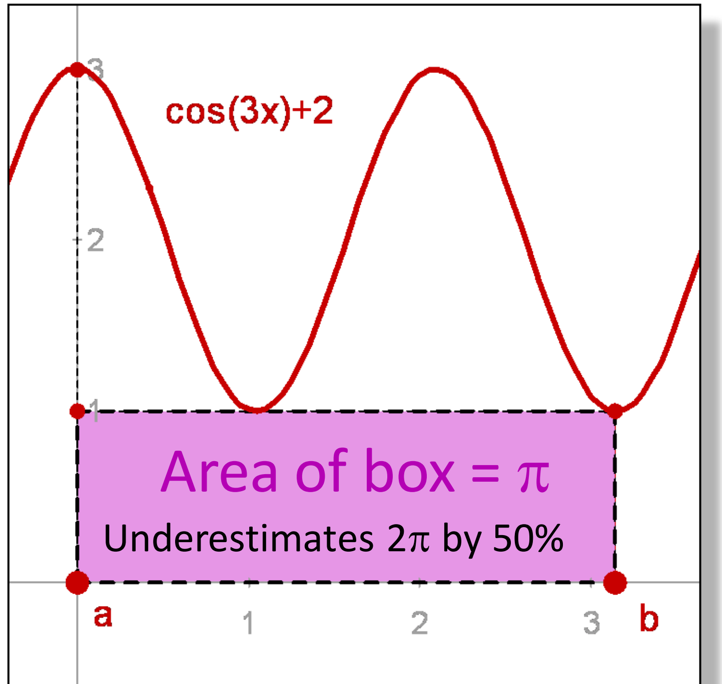
We will attempt this by a series of approximations.



Drawing Boxes

Lecture20-AreaUnderCurve1.gx
Lecture20-AreaUnderCurve2.gx





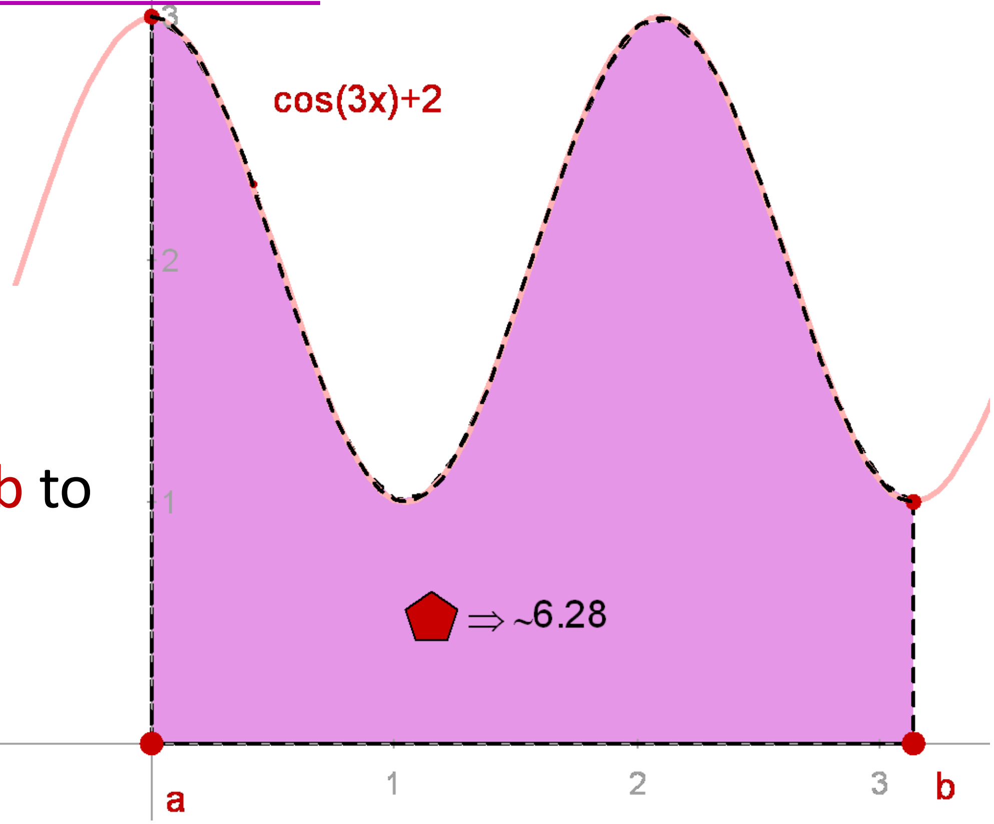
Letting $a = 0$ and $b = \pi$ we can draw simple boxes that overestimate and underestimate the area under the curve.

Averaging the results in this case gives 2π, the exact result!

Area Under Curve: Exact Solution

Lecture20-AreaUnderCurve.gx

For the general case we integrate the function between the limits a and b to obtain the area under the curve.



$$\int_{x=0}^{x=\pi} \cos\ 3x\ + 2\ dx = \int_{x=0}^{x=\pi} \cos\ 3x\ dx + 2\int_{x=0}^{x=\pi} dx =$$

$$\frac{1}{3}\left[\sin\ 3x\ \right]_{x=0}^{x=\pi} + 2\ x\ \Big|_{x=0}^{x=\pi} = \frac{1}{3}\left[\sin\ 3\pi\ - \sin\ 3\cdot 0\ \right] + 2\ \pi - 0\ = 2\pi$$

Computing Area in Maxima™

Lecture20-AreaUnderCurve.wxm

```
integrate(cos(3*x)+2, x);
```

$$\frac{\sin(3x)}{3}+2x$$

```
integrate(cos(3*x)+2, x, 0,%pi);
```

2 %pi

```
float(%);
```

6.283185307179586

1) Integrate the function to find the general result.

2) Integrate the function between x = 0 and x = π to obtain the area under the curve.

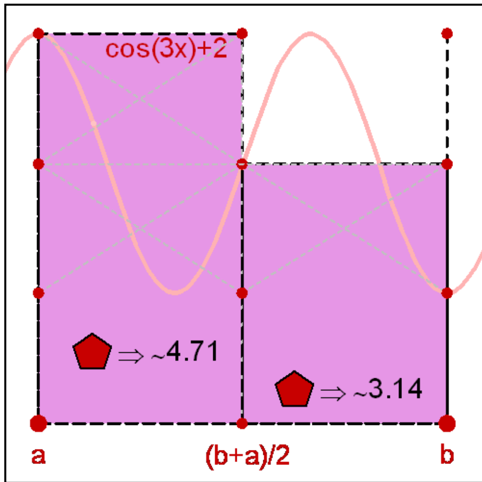
3) Evaluate the result numerically to obtain a decimal number. Is this 2π?

Notes:

- In Step 1, Maxima did not furnish a constant of integration!
- In Step 2, No constant of integration was needed… why?

Successive Approximations

Lecture20-AreaUnderCurve4a-c.gx



Overestimates by 25%

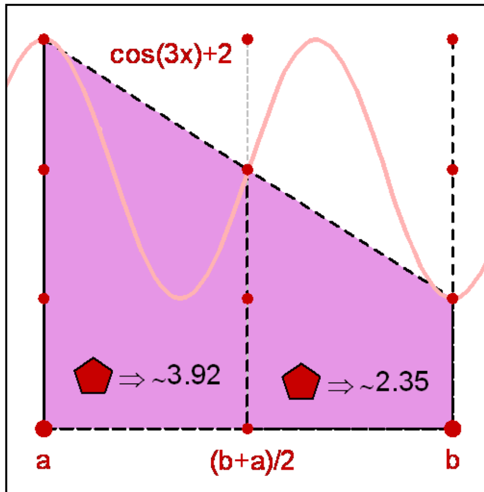
Area = 7.85

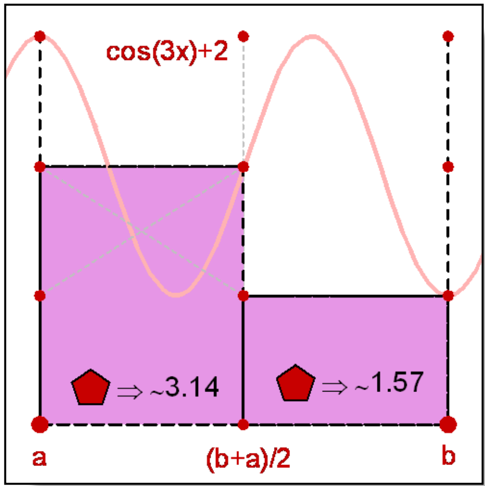
When we average the error is halved!

The **error** depends on the function and the positions of a and b.

Average is Trapezoid Estimates within 1%

Area = 6.29





Underestimates by 25%

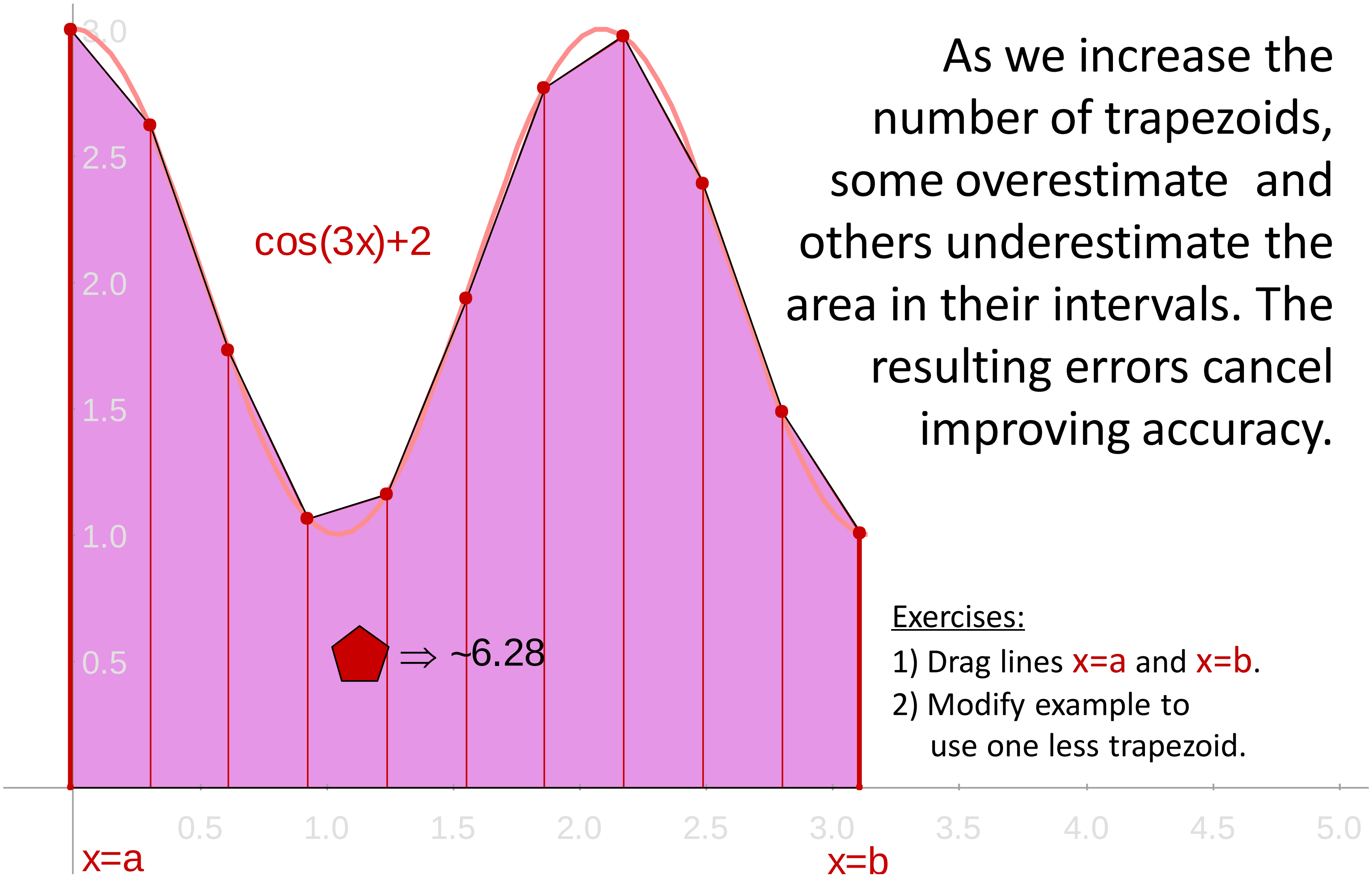
Area = 4.71

"**Convergence**" says how fast we approach the exact answer as we increase the number of boxes.

We can increase the number of boxes. Averaging overestimate and underestimate is equivalent to using trapezoids!

Trapezoidal Approximation: n=10

Lecture20-AreaUnderCurve5.gx



As we increase the number of trapezoids, some overestimate and others underestimate the area in their intervals. The resulting errors cancel improving accuracy.

Exercises:

1) Drag lines x=a and x=b.
2) Modify example to use one less trapezoid.

Units of Area

Lecture20-AreaUnderCurve6.gx

So far, units of measure have been absent or dimensionless.

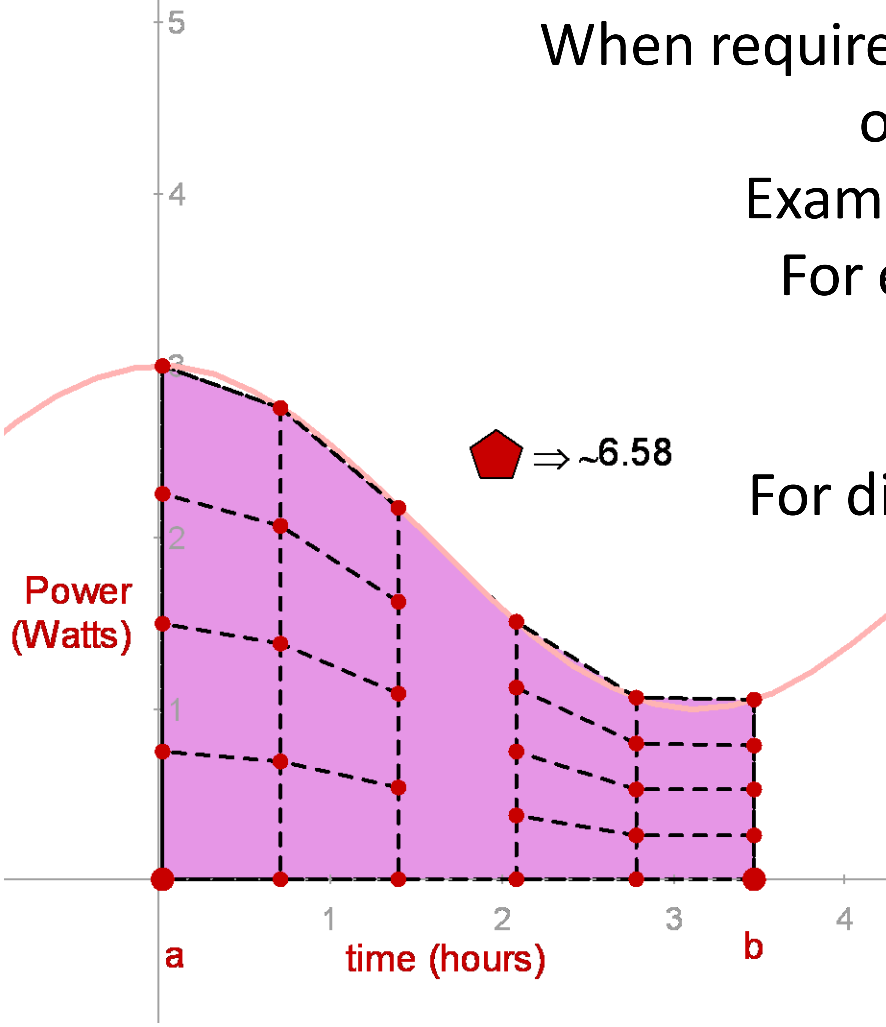
When required, units are simply the product of the x-axis units $\times$ y-axis units.

Examples:

For energy: $Watts \times hours = Watt \cdot hours$.

For carpet: $square\ yards\ or\ m^2$.

For distance traveled: $\dfrac{feet}{sec} \cdot sec = feet$.



Exercise:

1) Open the example. Drag a an b.
2) Lock a at 0.0 and b at 3.1416
3) Deduce the units rule for derivatives.

Area by Left Riemann Sum

Lecture20-AreaUnderCurve7.gx

From the left we can estimate area using the Riemann Sum. The width of each strip:

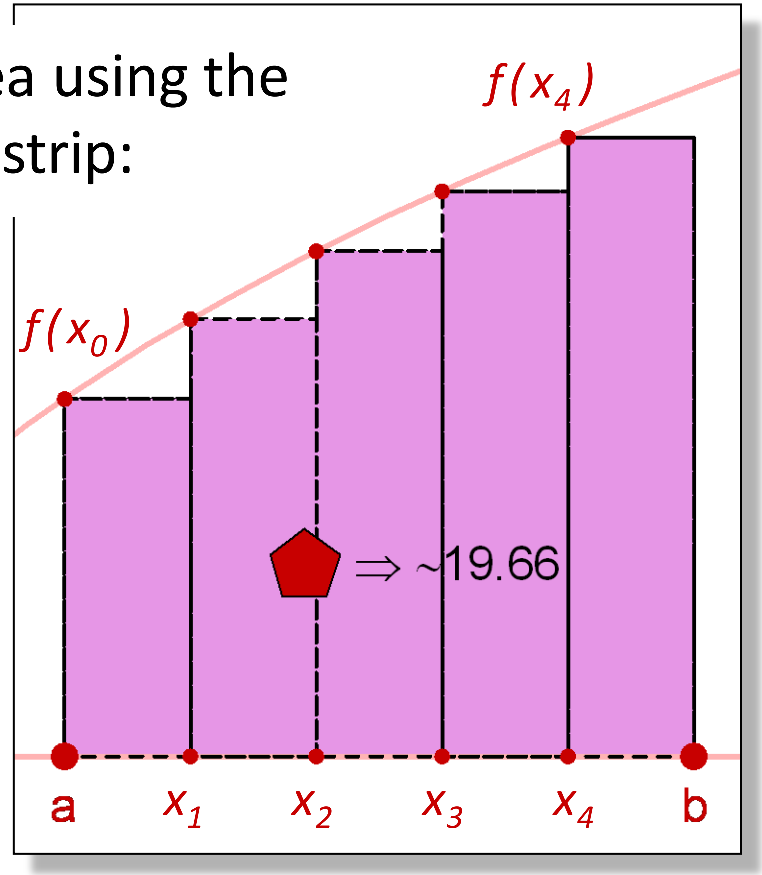
$$\Delta x = \frac{(b-a)}{n}$$

is multiplied by the height of each strip:

$$f(x_i)$$

and combined to compute:

$$Area = \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x$$



Exercises:

1) Drag a and b , then reset them to 2 and 7 respectively.
2) What is the value of ***i*** when x_i = a, when x_i = b?
3) Extend the example from n=5 to n=6.

Area by Right Riemann Sum

Lecture20-AreaUnderCurve8.gx

From the right we can also estimate area using the Riemann Sum. The strip width remains:

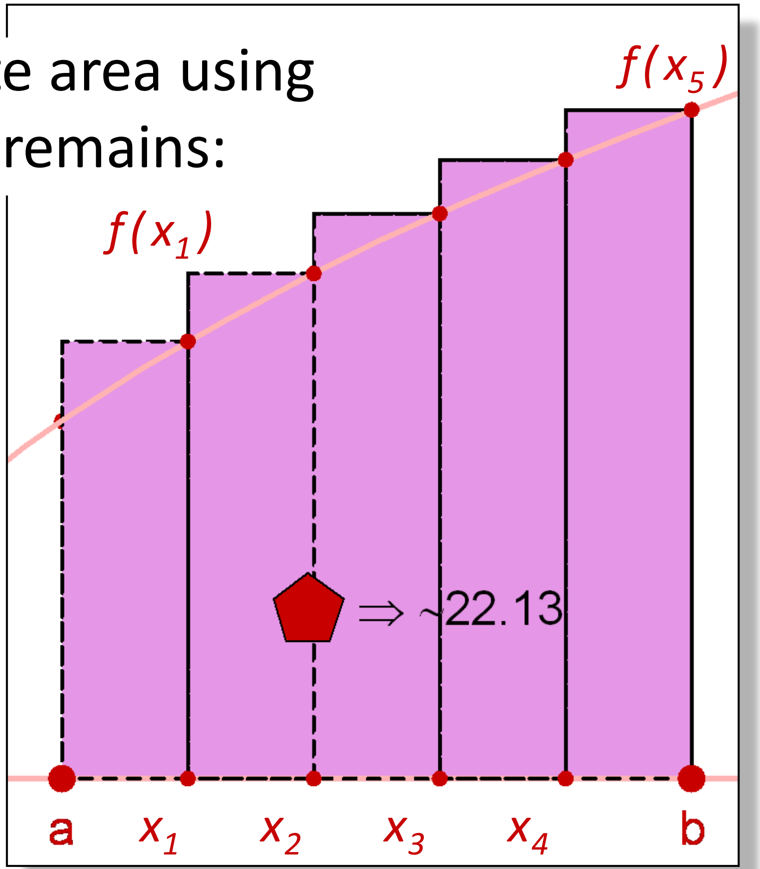
$$\Delta x = \frac{(b-a)}{n}$$

as does the strip height:

$$f(x_i)$$

The products are summed to give:

$$Area = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$



Exercises:

1) How is the index i different in the Left and right Riemann Sums?
2) Write out each term of the summation for the n=5 case.
3) Extend the example from n=5 to n=6.

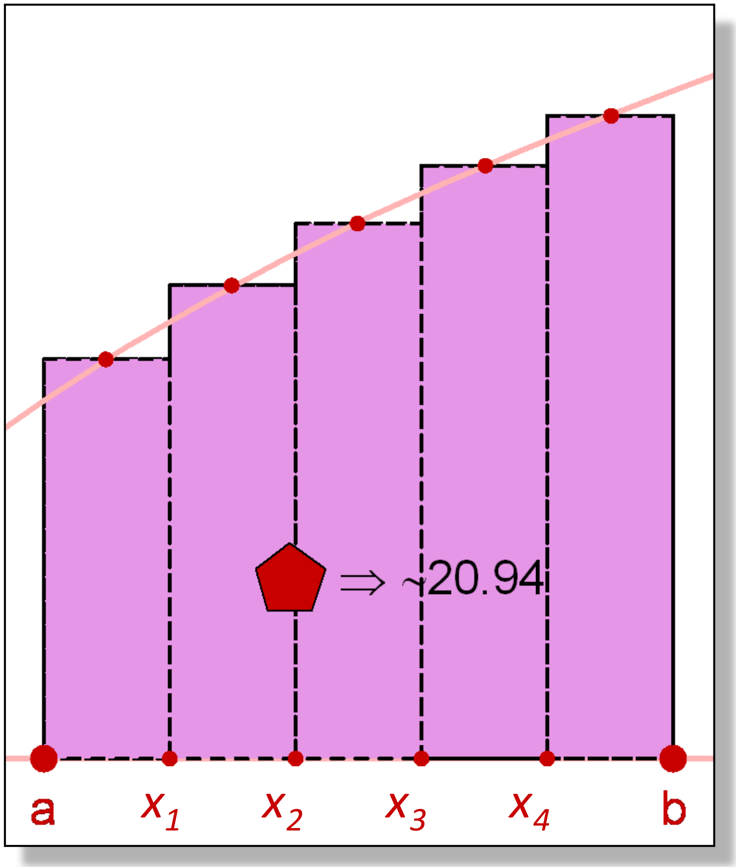
Area by Mean Riemann Sum

Lecture20-AreaUnderCurve9.gx

We can combine the left and right Riemann sums to provide an estimate that converges more quickly to the value of the area under the curve:

$$f(\overline{x_i}) = \frac{f(x_i) + f(x_{i+1})}{2}$$

$$Area = \sum_{i=0}^{n-1} f(\overline{x_i}) \cdot \Delta x$$



Exercises:

1) Drag a and b.
2) Write out each term of the summation for the n=5 case.
3) Extend the example from n=5 to n=6.

Limit of Riemann Sum

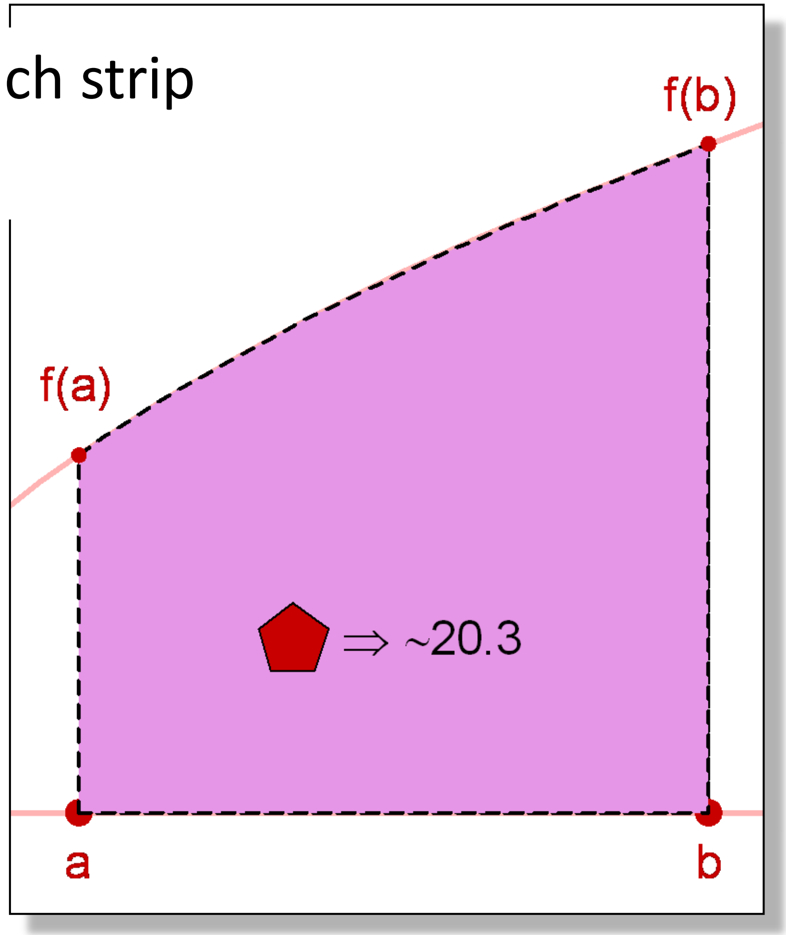
Lecture20-AreaUnderCurve10.gx

What happens to width of the each strip as we take the limit as n → ∞ of:

$$\Delta x = \frac{(b-a)}{n} \qquad ?$$

It can be shown that:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\cdot\left(\frac{b-a}{n}\right) = \int_a^b f(x)\cdot dx$$



Exercises:

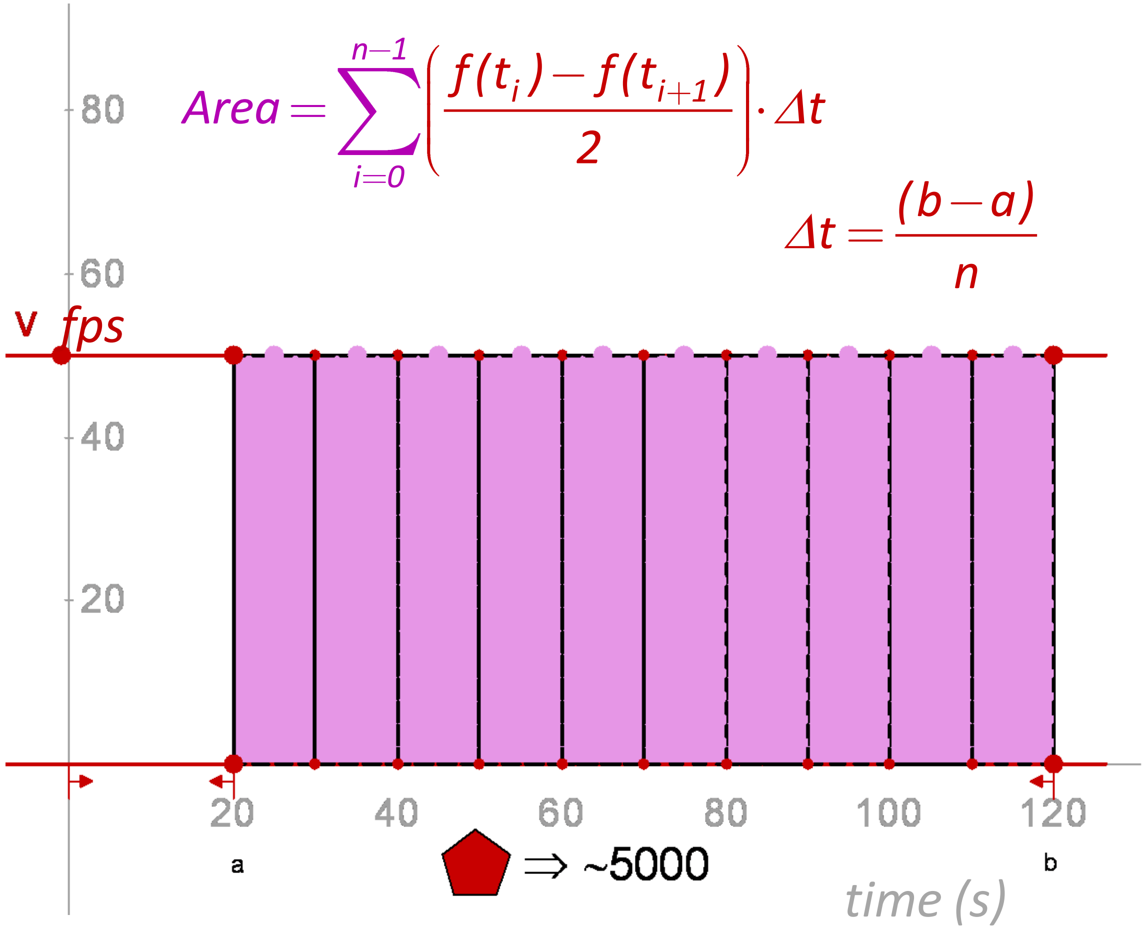
1) Drag a and b.
2) Change the function from $\sqrt{x}$ to x^2 and find the area under the curve between a = 2 and b = 7.

Integrating Constant Velocity

Lecture20-IntegratingVelocity.gx

Integrating constant velocity with respect to time gives distance.

An object traveling 50 feet per second for 100 seconds will travel 5000 feet.



Exercises:

1) What is n?
2) Drag v vertically to discover a new distance traveled in 100 seconds.
3) Change the time interval boundaries a and b change the duration.

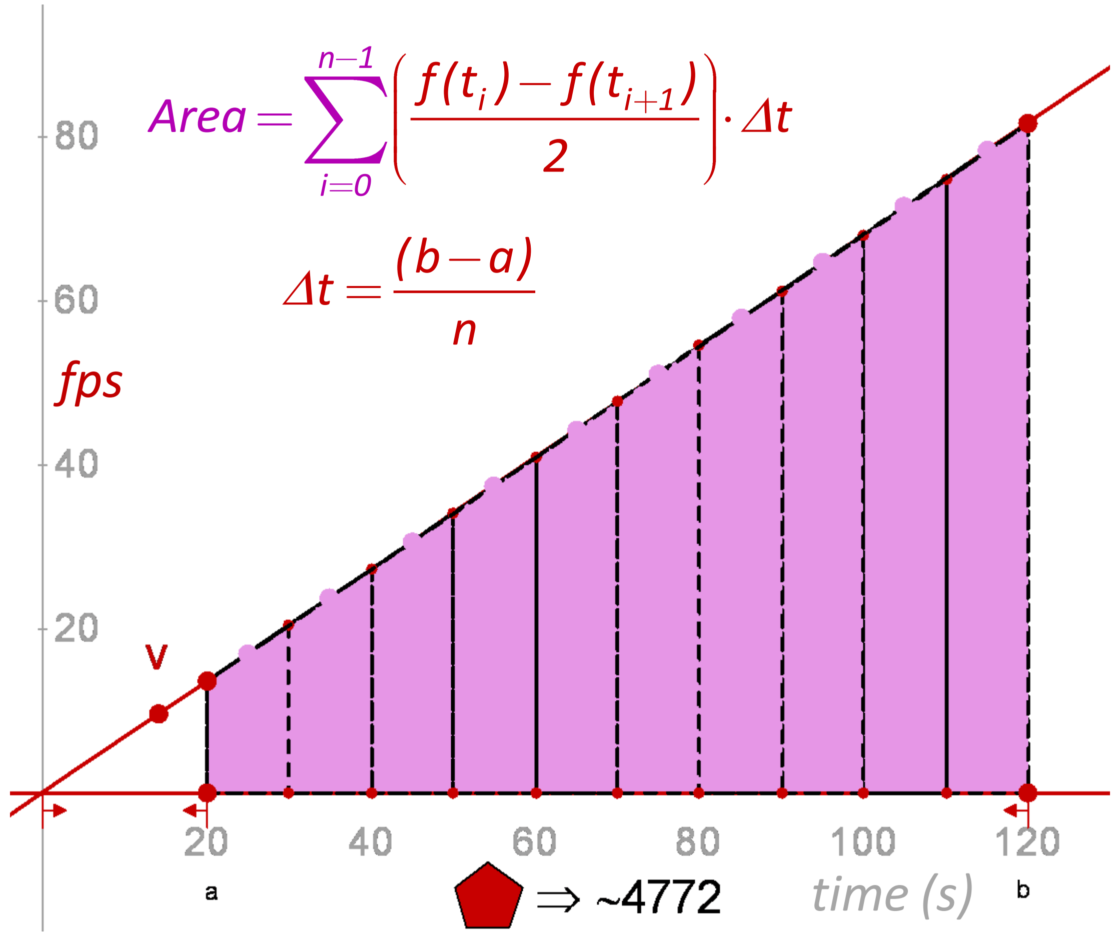
Integrating Linear Velocity

Lecture20-IntegratingLinearVelocity.gx

Integrating linear velocity with respect to time also gives distance.

Computation of the integral is identical.

The function is *sampled* at n=10 discrete times t_i.



Exercises:

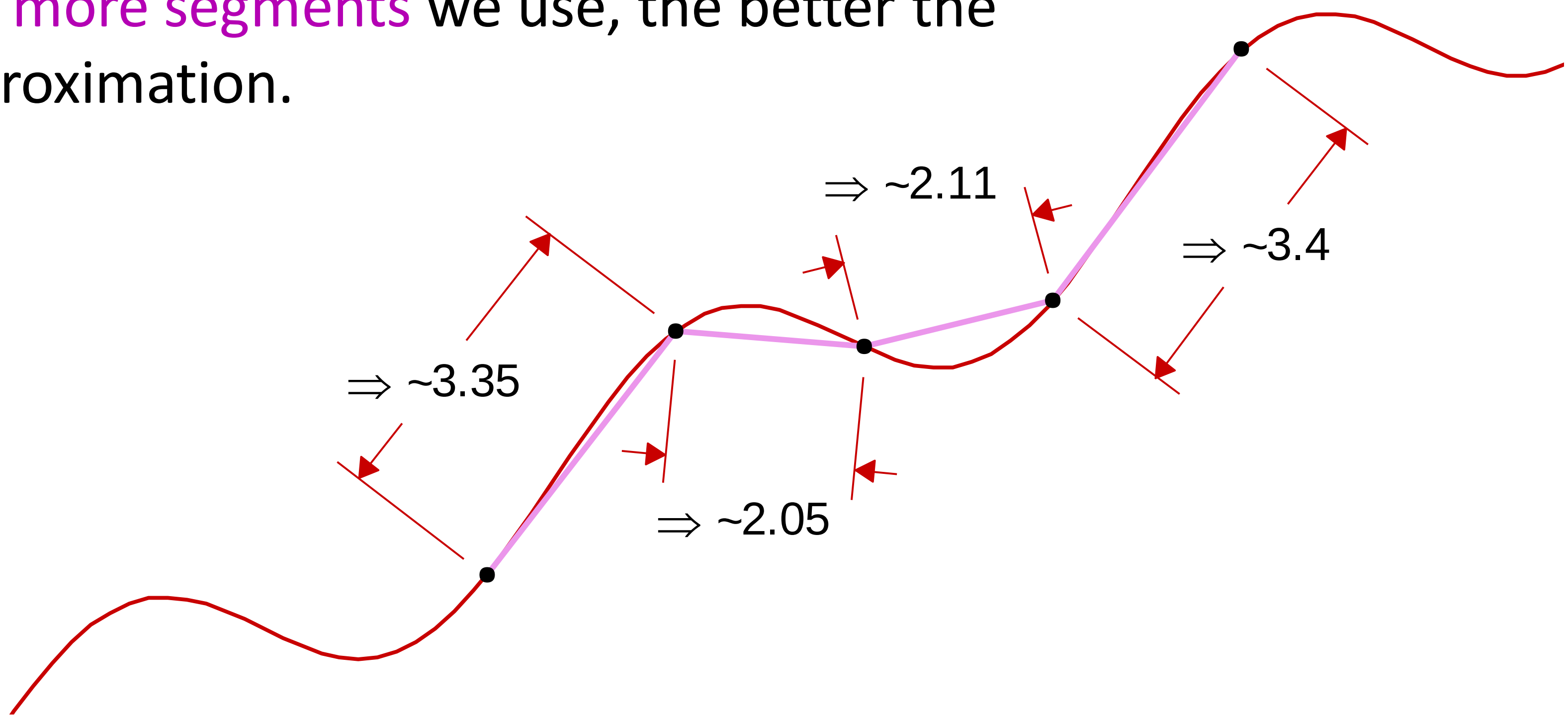
1) Drag v vertically to discover a new distance traveled between 20 and 100 seconds.
2) Change the time interval boundaries a and b change the duration.

Integrating Arc Length

Lecture20-IntegratingArcLength.gx

Just as we added strips of area together to find area, we can add segments of a curve to obtain the length of the curve.

The more segments we use, the better the approximation.

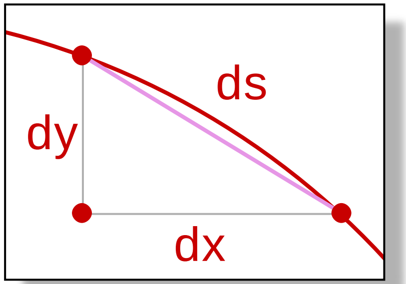


Integrating Arc Length

Lecture20-ArcLengthDiagram.gx

To derive a formula for integrating arc length we start with an infinitesimal arc length ds, computed via the Pythagorean theorem:

$$ds^2 = dx^2 + dy^2$$



$$\frac{ds^2}{dx^2} = \frac{dx^2}{dx^2} + \frac{dy^2}{dx^2} \quad \rightarrow \quad \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \rightarrow \quad ds = \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx$$

$$s = \int_a^b ds \quad \rightarrow \quad s = \int_a^b \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx$$

Integrating Arc Length

Lecture20-ArcLengthCircle.wxm
Lecture20-ArcLengthCircle.gx

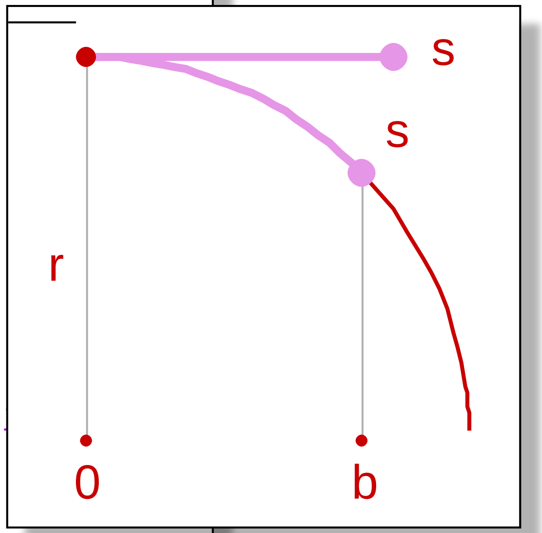
With all this machinery, it is a good idea to test against a known case. A circle is a good choice for our first arc length integral.

```
(%i1) diff(sqrt(r^2-x^2), x);
```

(%o1) $-\frac{x}{\sqrt{r^2-x^2}}$

```
(%i2) integrate(sqrt(1+%^2),x,0,b);
Is  r  positive or negative?positive;
Is  b  positive, negative, or zero?pos
```

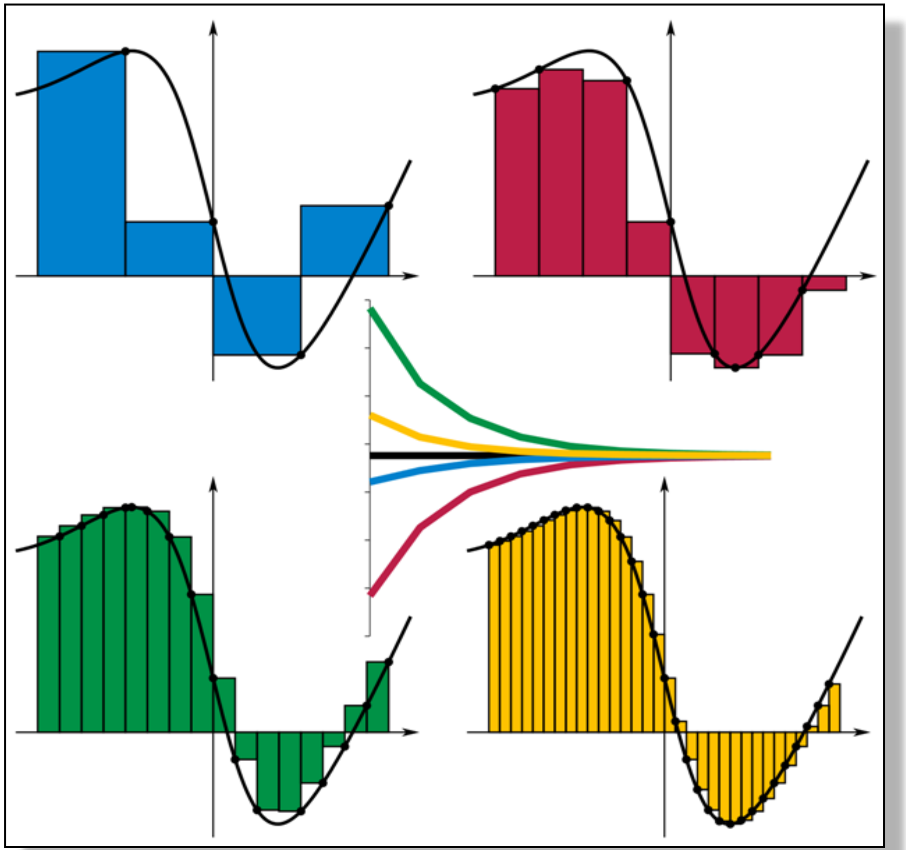
(%o2) $\operatorname{asin}\left(\frac{b}{r}\right) r$



Few functions can be integrated in closed form, so numerical methods are often used.

Numerical Convergence

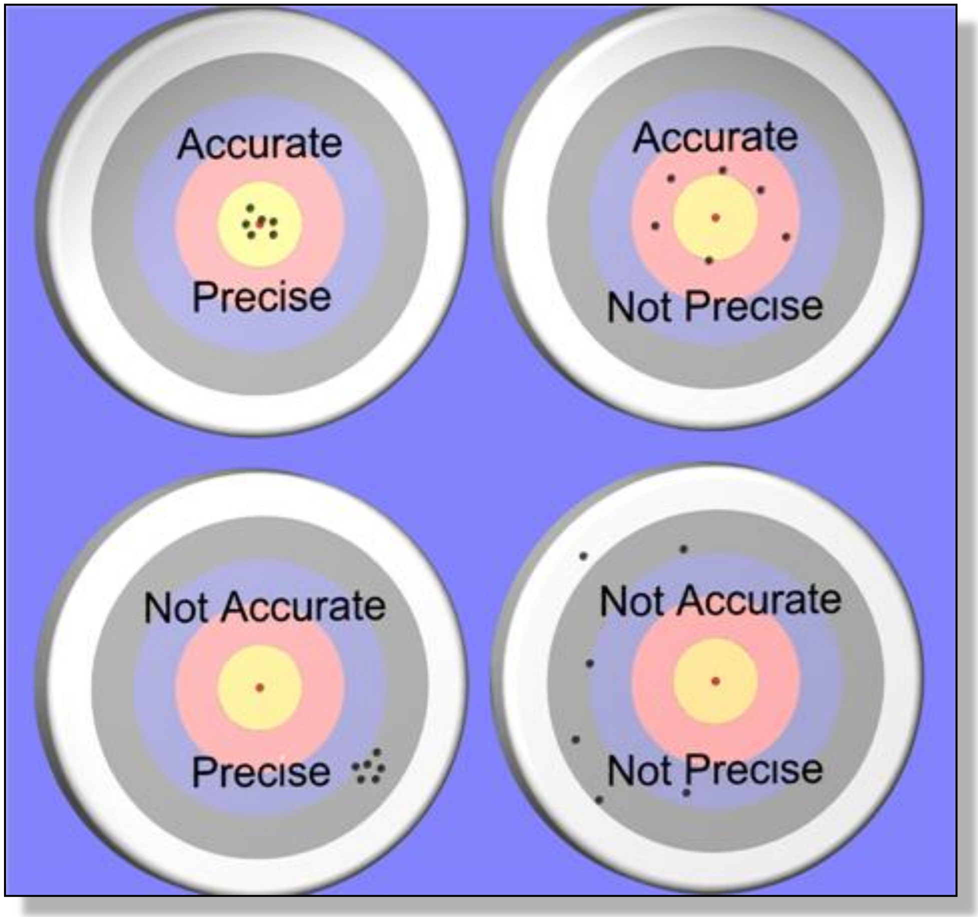
When we perform numerical integration, the number of pieces we add together makes a difference on the accuracy and precision of the solution.



Exercises:

Look up the definitions of:

- *floating point overflow*
- *floating point underflow*
- *round-off error*
- *truncation-error*
- *numerical instability*


Accurate
Precise
Accurate
Not Precise
Not Accurate
Precise
Not Accurate
Not Precise

End