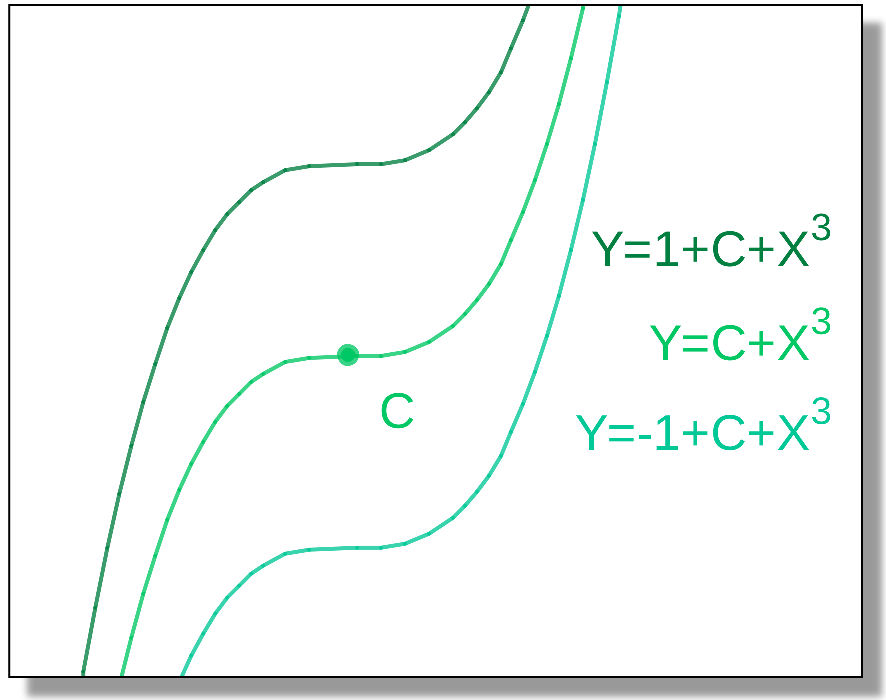
Learning Calculus With Geometry Expressions™

by L. Van Warren



- Three Antiderivatives of $Y=X^2$

Lecture 19: Antiderivatives & The Indefinite Integral

Chapter 5: Integration

LECTURE	TOPIC
19	**ANTIDERIVATIVES**
20	INTEGRATION: AREA AND DISTANCE
21	THE DEFINITE INTEGRAL
22	FUNDAMENTAL THEOREM OF CALCULUS

Inspiration

Alonzo Church 1903 - 1995
Princeton / UCLA Mathematician

- Lambda Calculus
  the calculus of anonymous functions.

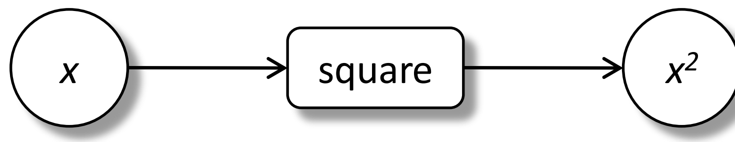
- Church-Turing Thesis

- Proof that First Order Logic
  is Undecideable.

- Influenced Functional Languages
  like Lisp, Python and Haskell

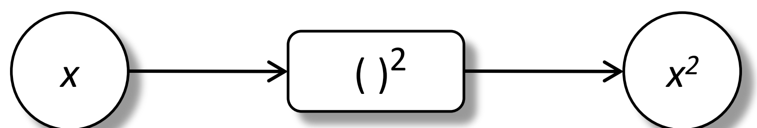
Anonymous Functions:

Up to now we have often used functions that have names. For example, $f(x) = x$, says that the function named f maps its input x to its output with no change. Another, named, *square(x)*, multiplies its input by itself and outputs the result, x^2.



It is possible to accomplish the same process without naming the functions. Instead we write:

$$x \rightarrow x^2$$



This notion, due to Church, can be greatly amplified and extended. For further information see the [wiki].

Undo, Inverse, Unmap:

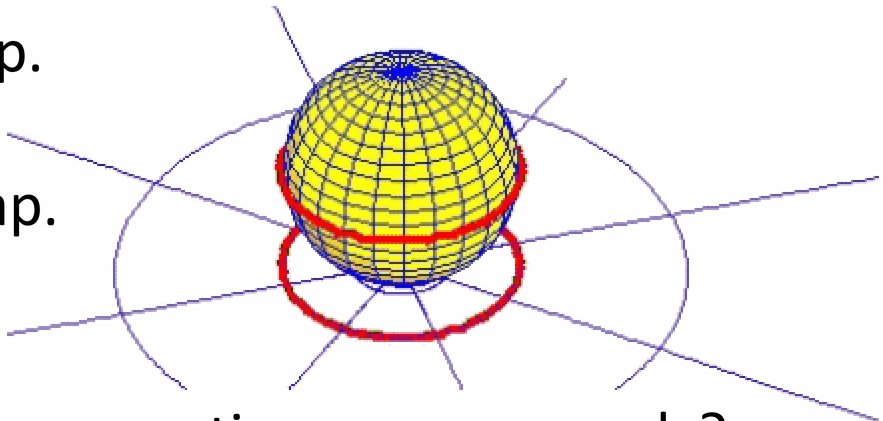
The Inverse of a function "undoes" the function:

$$f^{-1}(f(x)) = x \quad \text{AND} \quad f(f^{-1}(x)) = x$$

A Map is a pairing or connection between values of x and f(x).

The Inverse "undoes" the map.

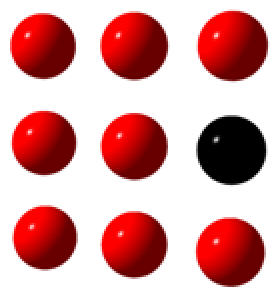
The Inverse "unmaps" the map.



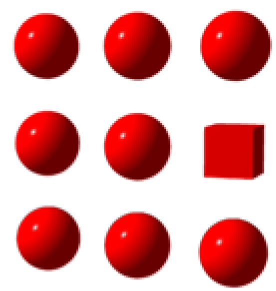
How would you represent these equations anonymously?

Uniqueness

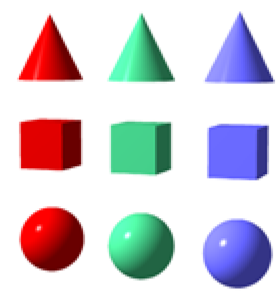
Unique
Color



Unique
Shape



Unique
Combinations



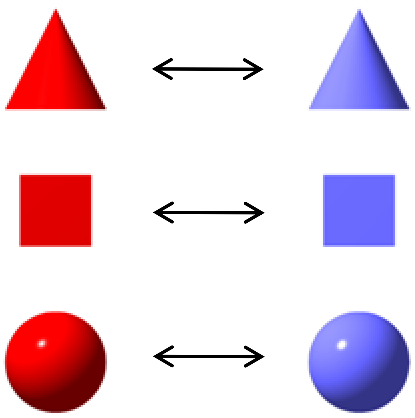
Uniqueness Implies Invertibility

If only one output value exists for any input value we say the relation is Unique.

Unique relations have one x value for every f(x) value.

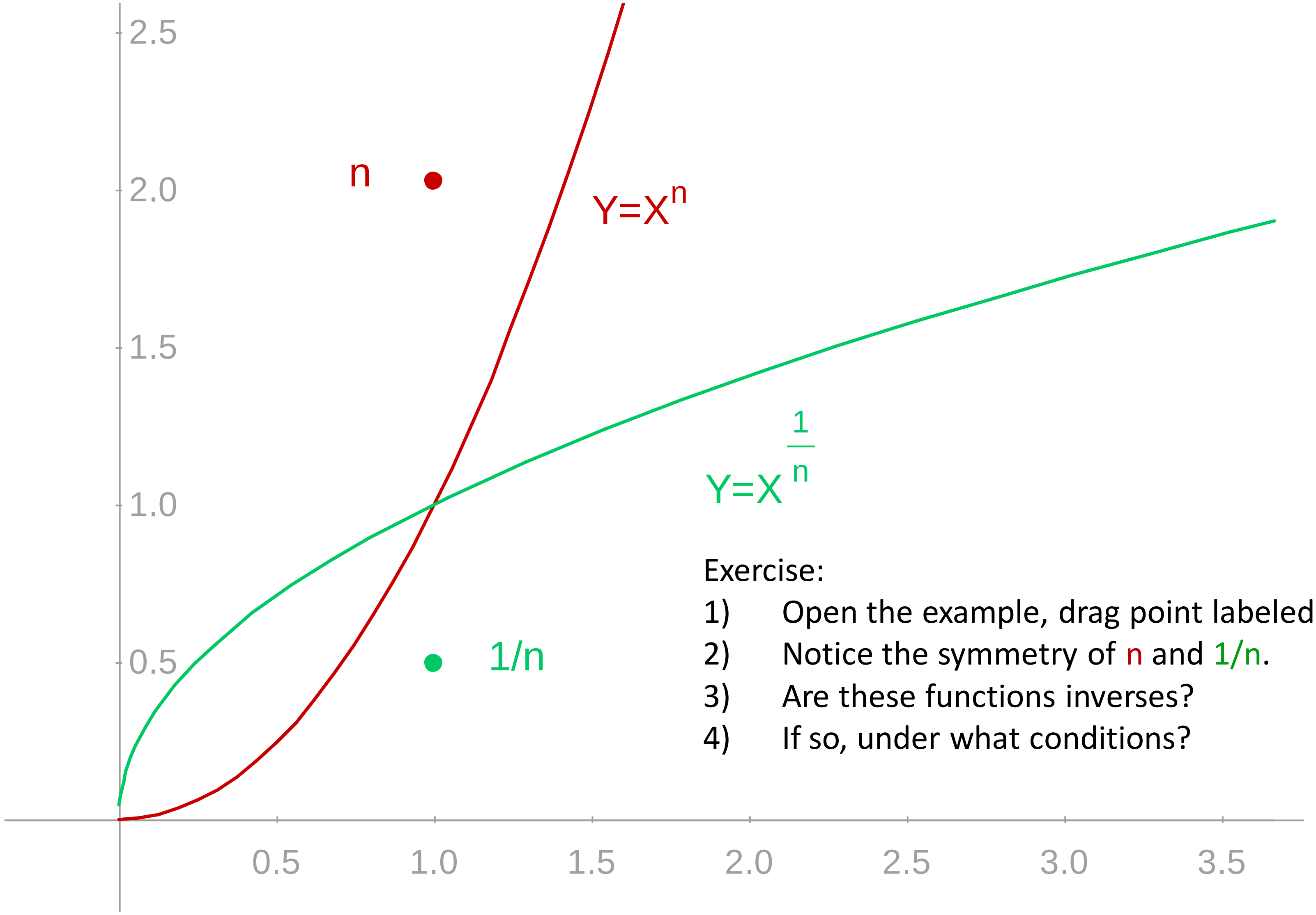
Uniqueness implies Invertibility.

This means we can undo the operation and recover the original value.



Lecture19-ACertainInverse.gx

Reviewing a Certain Inverse:

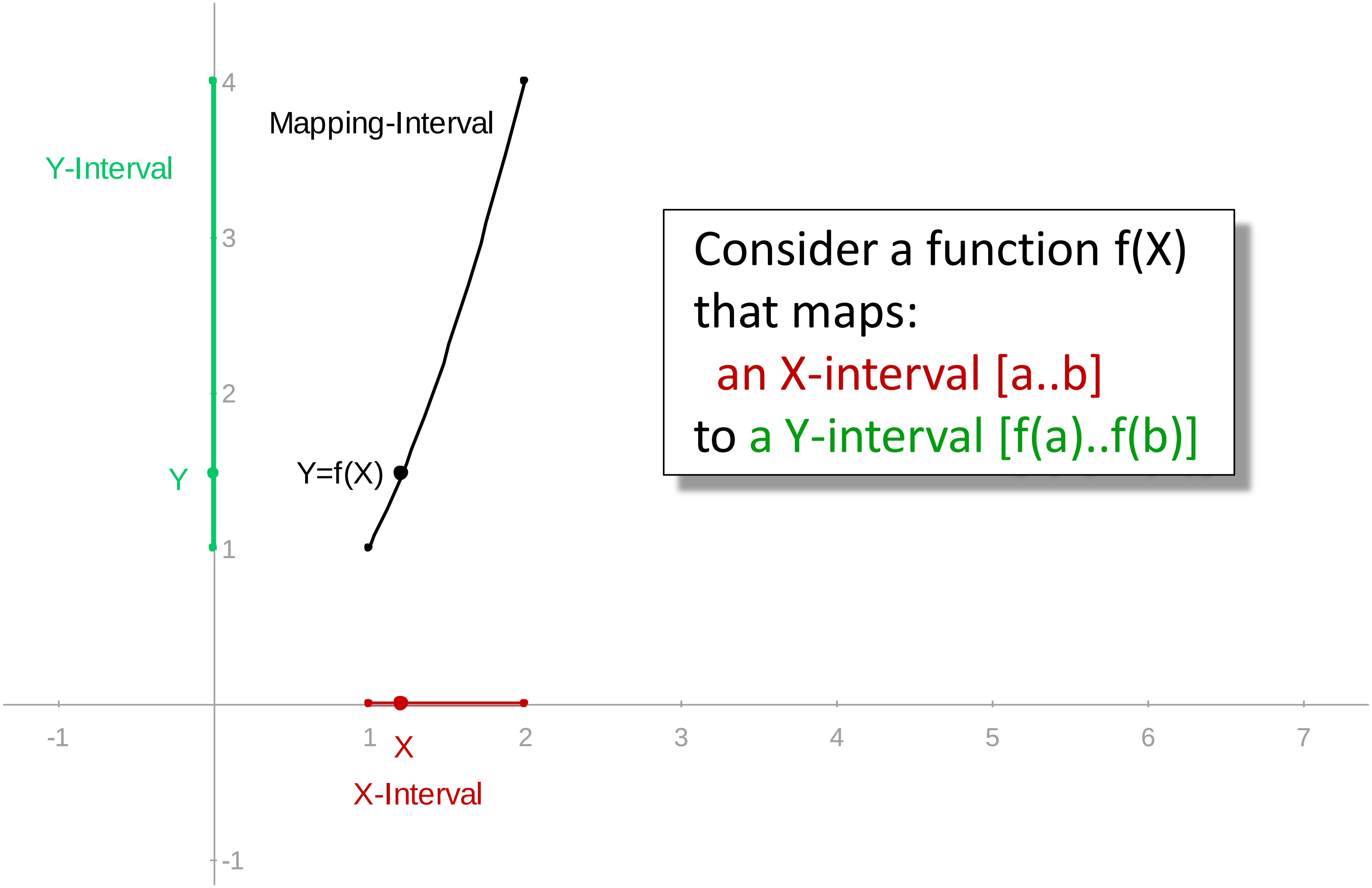


Exercise:

1) Open the example, drag point labeled n.
2) Notice the symmetry of n and 1/n.
3) Are these functions inverses?
4) If so, under what conditions?

Dummy Variables:

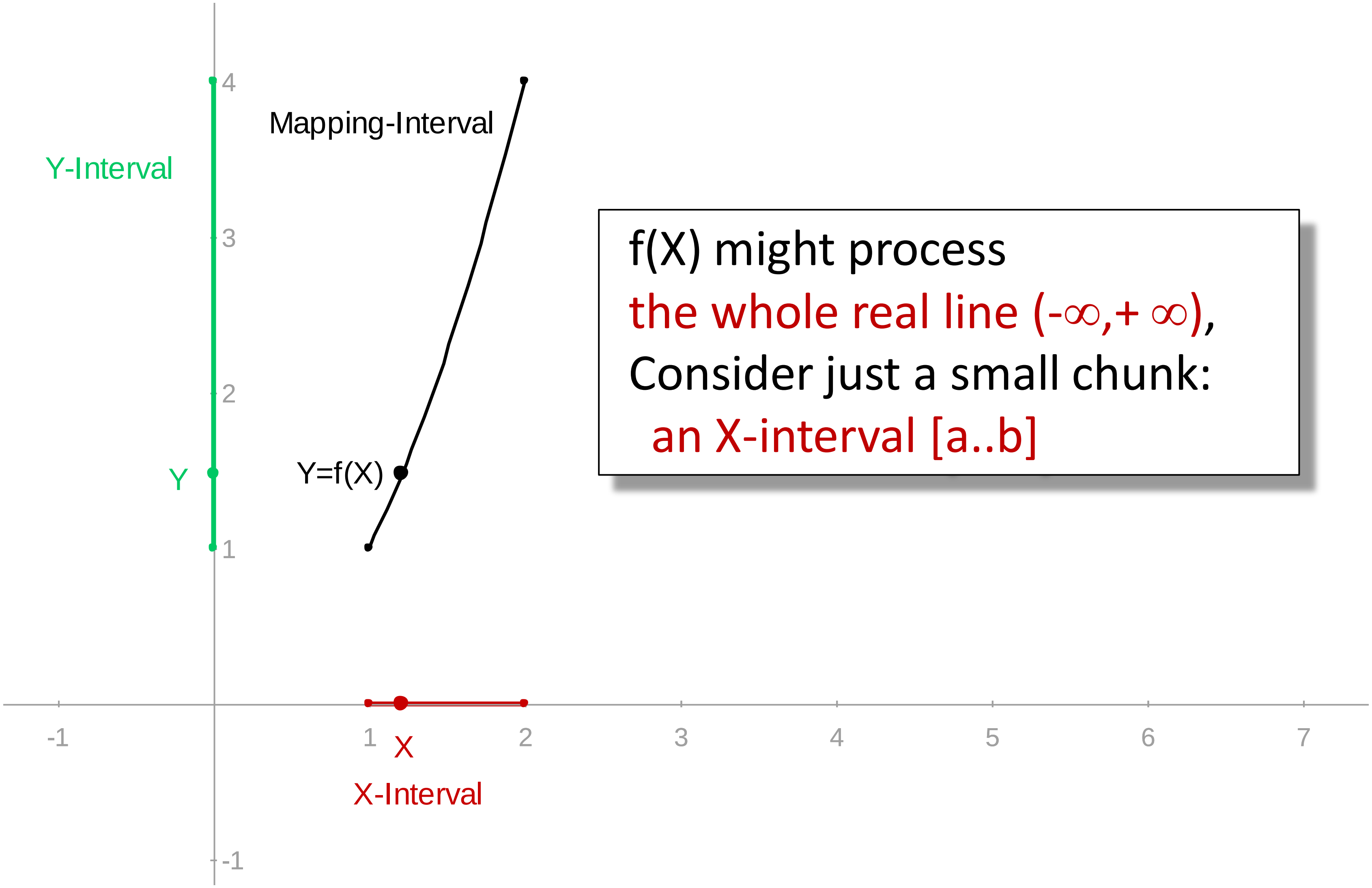
Lecture19-MappingInterval.gx



Consider a function f(X) that maps:
an X-interval [a..b]
to a Y-interval [f(a)..f(b)]

Dummy Variables:

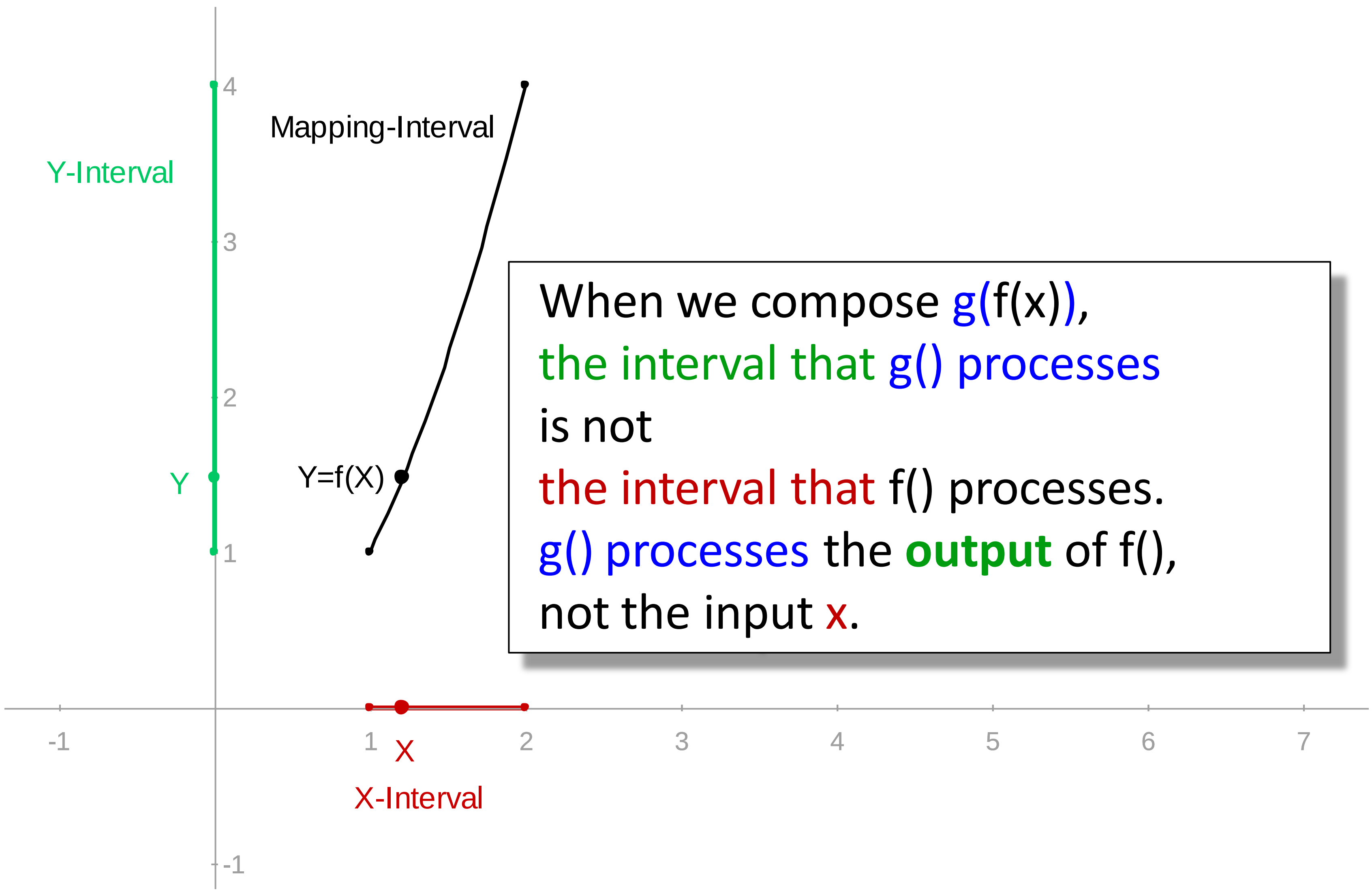
Lecture19-MappingInterval.gx



f(X) might process
the whole real line $(-\infty, +\infty)$,
Consider just a small chunk:
an X-interval [a..b]

Dummy Variables:

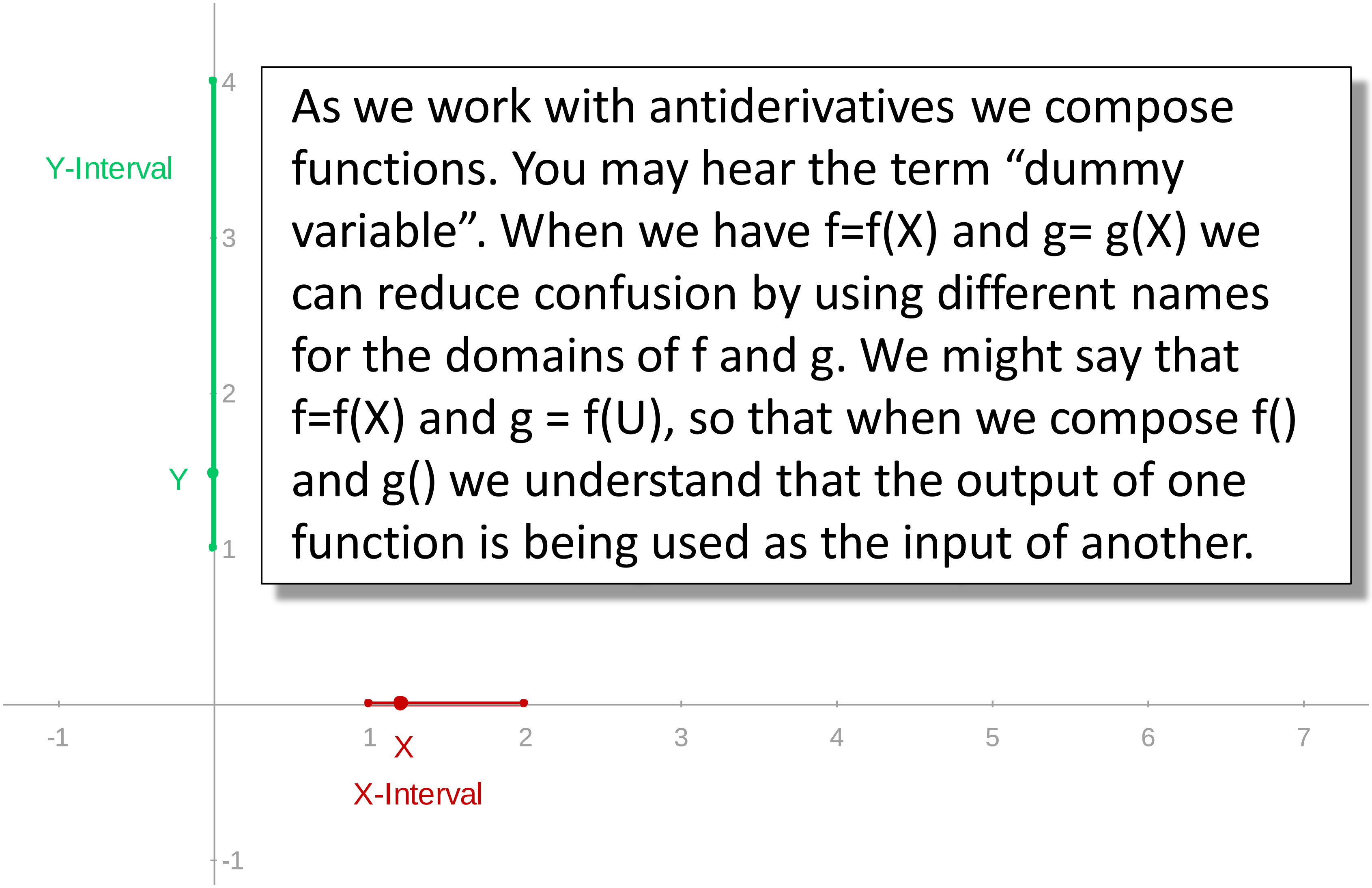
Lecture19-MappingInterval.gx



When we compose g(f(x)),
the interval that g() processes
is not
the interval that f() processes.
g() processes the **output** of f(),
not the input x.

Lecture19-MappingInterval.gx

Dummy Variables:



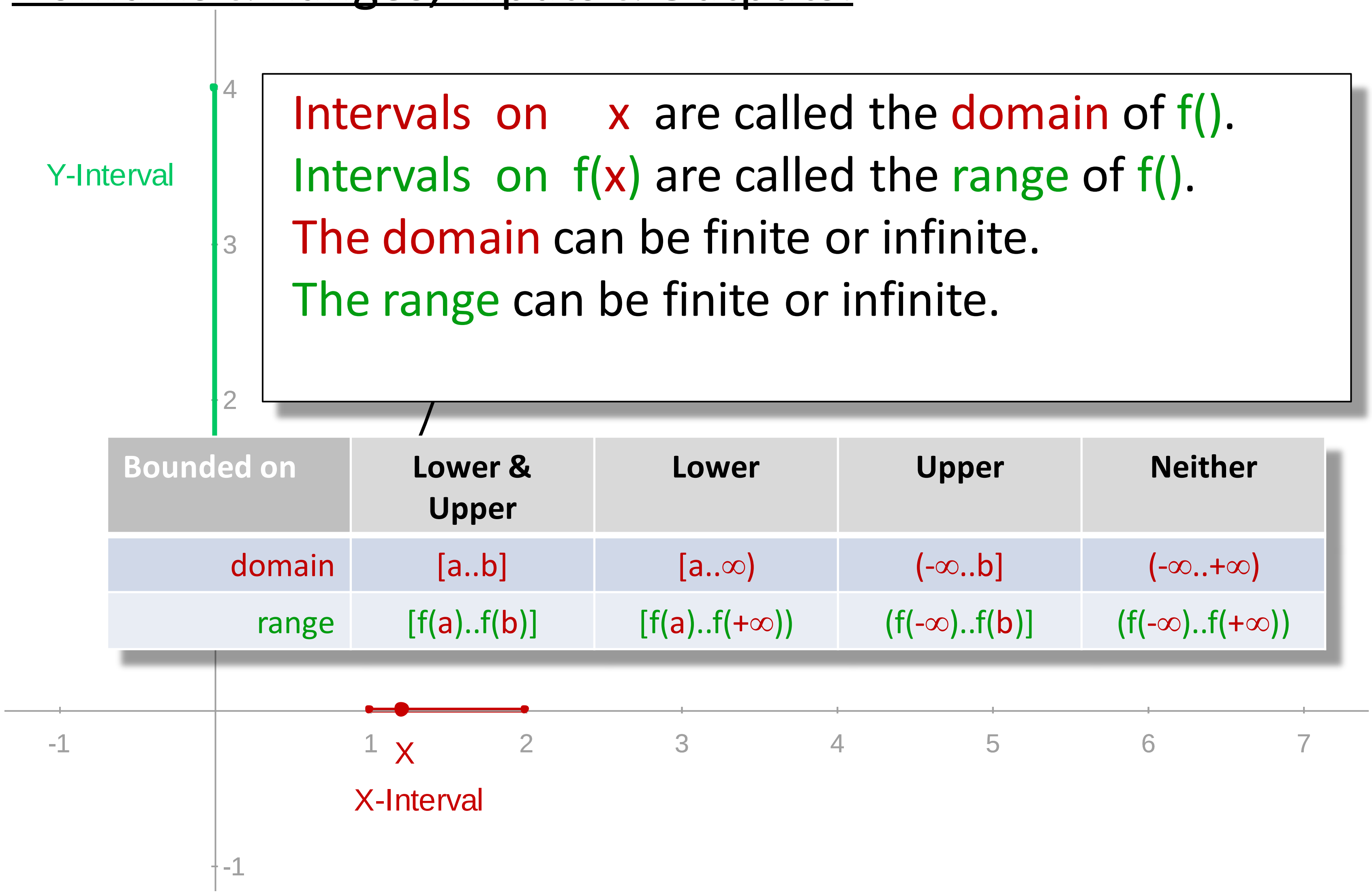
As we work with antiderivatives we compose functions. You may hear the term “dummy variable”. When we have f=f(X) and g= g(X) we can reduce confusion by using different names for the domains of f and g. We might say that f=f(X) and g = f(U), so that when we compose f() and g() we understand that the output of one function is being used as the input of another.

Lecture19-MappingInterval.gx

Domains & Ranges, Inputs & Outputs:

Intervals on x are called the domain of f().
Intervals on f(x) are called the range of f().
The domain can be finite or infinite.
The range can be finite or infinite.

Bounded on	Lower & Upper	Lower	Upper	Neither
domain	[a..b]	[a..∞)	(-∞..b]	(-∞..+∞)
range	[f(a)..f(b)]	[f(a)..f(+∞))	(f(-∞)..f(b)]	(f(-∞)..f(+∞))



The Derivative and Antiderivative:

Having discussed inverse functions, consider the inverse of the derivative operator, that "undoes" the derivative. This inverse is called the antiderivative. Another name for the antiderivative of f(x) is the integral of f(x).

If:

$$\frac{d}{dx} f(x) = g(x)$$

Then:

g(x) is the derivative of f(x) and
f(x) is the antiderivative of g(x)

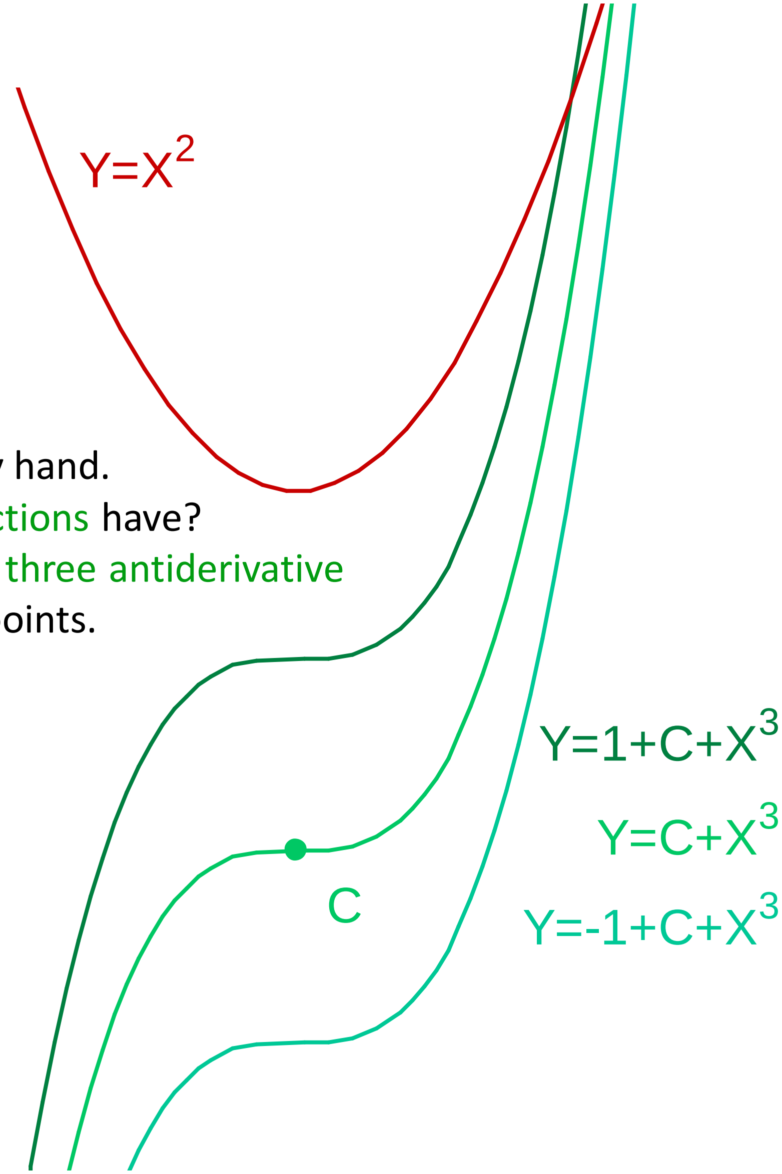
And we write:

$$\int g(x)dx = f(x) + c$$

The antiderivative can also be called "undiff".

Lecture19-ACertainAntiderivative.gx

A Certain Antiderivative:



Exercises:

1) Open the example, drag point labeled C.
2) Differentiate each green cubic function by hand.
3) For large X, what values do the green functions have?
4) Intersect the red function with each of its three antiderivative functions, to discover three intersection points.

$$\int X^2\,dX = \frac{X^2}{3} + c$$

Differentiating in Reverse for Falling:

Previously we wrote the position of a falling object as:

$$y = f(t) = -\frac{1}{2}gt^2$$

Assuming "up" is positive, differentiating yields:

$$\frac{dy}{dt} = f'(t) = v = -gt$$

Differentiating again we are left like our friend here with the acceleration due to gravity:

$$\frac{d^2y}{dt^2} = f''(t) = a = -g$$

The minus sign reminds us that gravity pulls.

-Li Wei

The Antiderivative is Just the Integral:

To undo the previous derivatives, we integrate:

$$v(t) = \int (-g)\,dt = -g \int dt = -gt + v_0$$

Lather, rinse, repeat to obtain:

$$y(t) = \int (-gt + v_0)\,dt = -g\frac{t^2}{2} + v_0 t + y_0$$

We can arrange our coordinates so that the constants of integration v_0 and y_0 are zero as we saw in the original case *or* we can work a richer set of problems.

The value of the contants are just the velocity v_0, and position y_0, at time zero, t_0.

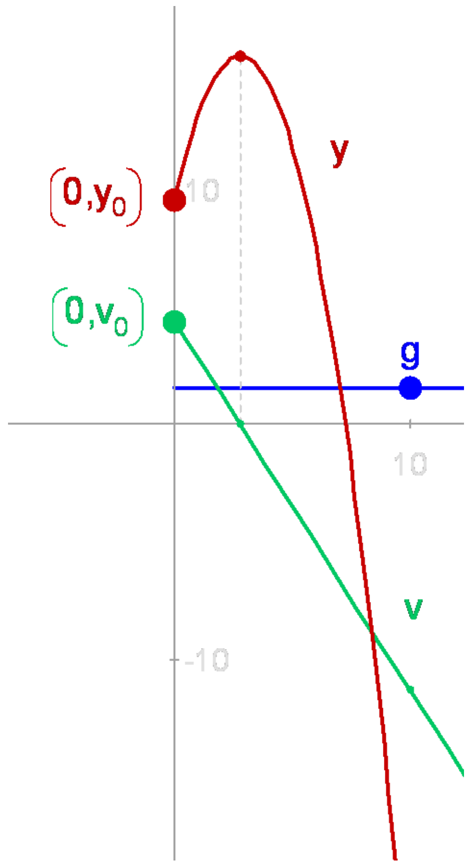
-Li Wei

Lecture19-Falling.gx

The Falling Integral:

Exercise:

1) Open the example, drag the two constants v_0 and y_0.
2) Change the gravitational acceleration g to match: the Moon (1.6 m/s), Earth (9.8 m/s) and Mars 3.7 (m/s). Add horizontal lines corresponding to these values of g. Note how sensitive the path of the object is to g.
3) The position of the object initially increases with time, why?
4) Where does max-min theory appear in this example?



$$y(t) = \int (-gt + v_0)dt = -g\frac{t^2}{2} + v_0 t + y_0$$

$$v(t) = \int (-g)dt = -g\int dt = -gt + v_0$$

The Falling Integral: Symbolically

Lecture19-Falling.wxm

```
v(t):=''(integrate(-g,t)+v[0]);
```

$$v(t):=v_0-g\,t$$

```
y(t):=''(integrate(v(t),t)+y[0]);
```

$$y(t):=-\frac{g\,t^2}{2}+v_0\,t+y_0$$

```
xCrit : rhs(first(solve(v(t),t)));
```

$$\frac{v_0}{g}$$

```
y(xCrit);
```

$$\frac{v_0^2}{2\,g}+y_0$$

1) Integrate acceleration to obtain velocity.
2) Integrate velocity to obtain position.
3) Find the critical x value.
4) Find the critical y value.

Defining Functions in Maxima™

Lecture19-UndiffSquare.wxm

```
Diff(f):=diff(f,x);
Diff(f):=diff(f, x)

Diff(x^2);
2 x

Undiff(f):=integrate(f,x);
Undiff(f):=integrate(f, x)

Undiff(2*x);
x^2
```

1) Define Diff(f) as the derivative of a function named f with respect to x.
2) Differentiate x^2 to obtain 2x
3) Define Undiff(f) as the integral of a function named f with respect to x.
4) Integrate 2x to obtain x^2.

Notes:

- "Case" counts: *Diff* is a different name than *diff* !
- Maxima did not furnish a constant of integration !

Antiderivative as Undiff:

We have used several notations for the same idea. We know:

$$\frac{d}{dx}\sin(x)=\cos(x)$$

We could just as well write:

$$Diff(\sin(x))=\cos(x)$$

and then take the antiderivative or “undiff” of both sides:

$$\begin{aligned} Undiff(Diff(\sin(x))) &= Undiff(\cos(x)) \\ \sin(x) &= Undiff(\cos(x)) \\ \sin(x) &= \int \cos(x)dx \end{aligned}$$

This is what we do with the *integrate* command in Maxima™.

Adding a Constant in Maxima™:

Lecture19-UndiffCos.wxm

```
Diff(f):=diff(f,x);
```
$\mathtt{Diff}(f):=\mathtt{diff}(f,x)$

```
Diff(sin(x));
```
$\cos(x)$

```
Undiff(f):=integrate(f,x)+C;
```
$\mathtt{Undiff}(f):=\mathtt{integrate}(f,x)+C$

```
Undiff(cos(x));
```
$C+\sin(x)$

1) Define Diff(f) as the derivative of a function named f with respect to x.
2) Differentiate sin(x) to obtain cos(x)
3) Define Undiff(f) as the integral of a function named f with respect to x PLUS a constant of integration.
4) Integrate cos(x) to obtain sin(x) + C.

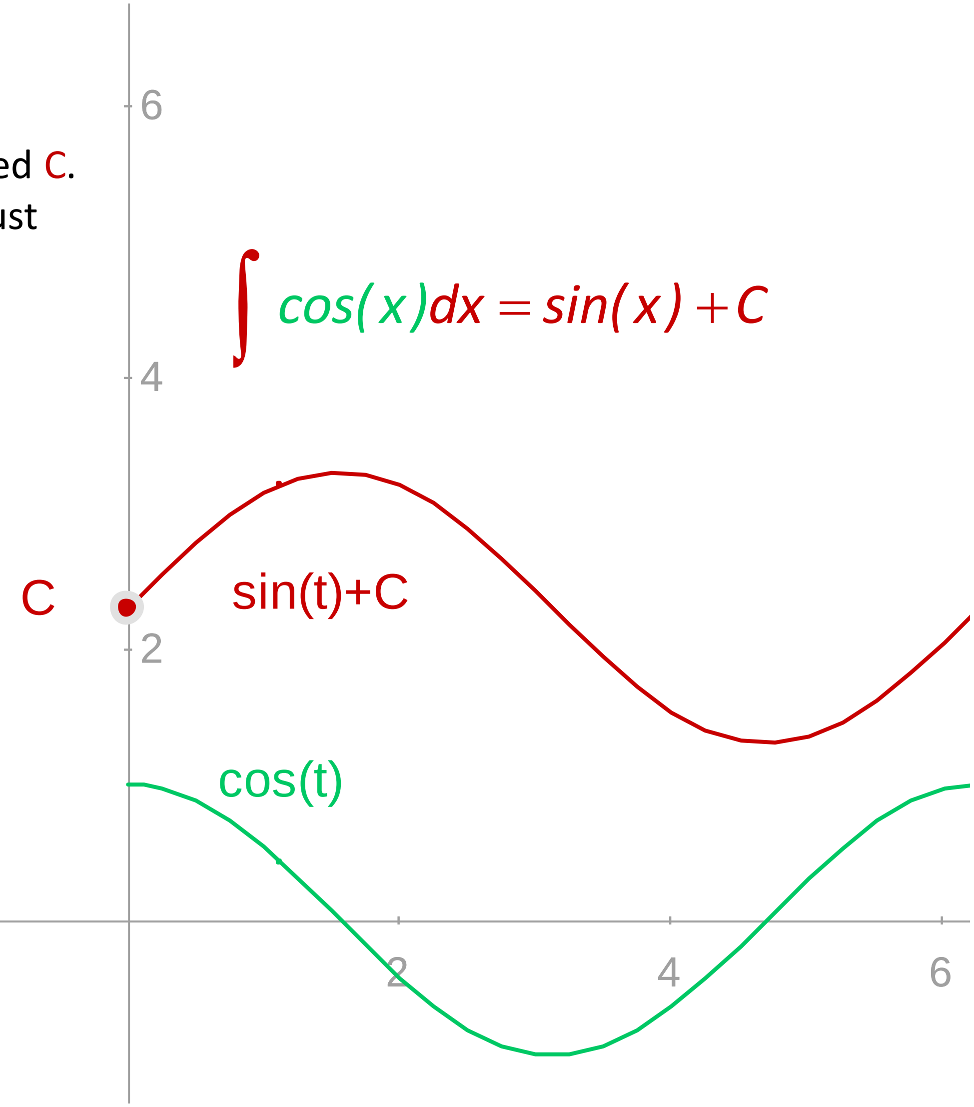
Diff Sine, Undiff Cosine in GX™:

Lecture19-UndiffCos.gx

Exercises:

1) Open the example, drag point labeled C.
2) Note that when we integrate we must always introduce a constant of integration.

By adding the constant we preserve the inverse.



The Antiderivative Power Law n = 1:

The power law states:

$$\frac{d}{dx}x^n = n \cdot x^{n-1}$$

For the special case of n = 1 we have:

$$\frac{d}{dx}x^1 = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1$$

We can write this as: $Diff(x) = 1$

And Undiff both sides:

$$Undiff(Diff(x)) = Undiff(1) = x + C$$

$$\int 1dx = \int dx \quad = x + C$$

-Li Wei

The Antiderivative Power Law n = 2:

For the case of n = 2 we have:

$$\frac{d}{dx}x^2 = 2\cdot x^{2-1} = 2\cdot x^1 = 2x$$

We can write this as:

$$\frac{Diff(x^2)}{2} = x$$

-Li Wei

And *Undiff* both sides:

$$Undiff\left(\frac{Diff(x^2)}{2}\right) = Undiff(x) = \frac{x^2}{2} + C$$

$$\int x\,dx = \frac{x^2}{2} + C$$

The Antiderivative Power Law n = 3:

For the case of n = 3 we have:

$$\frac{d}{dx}x^3 = 3\cdot x^{3-1} = 3x^2$$

We can write this as:

$$\frac{Diff(x^3)}{3} = x^2$$

And *Undiff* both sides:

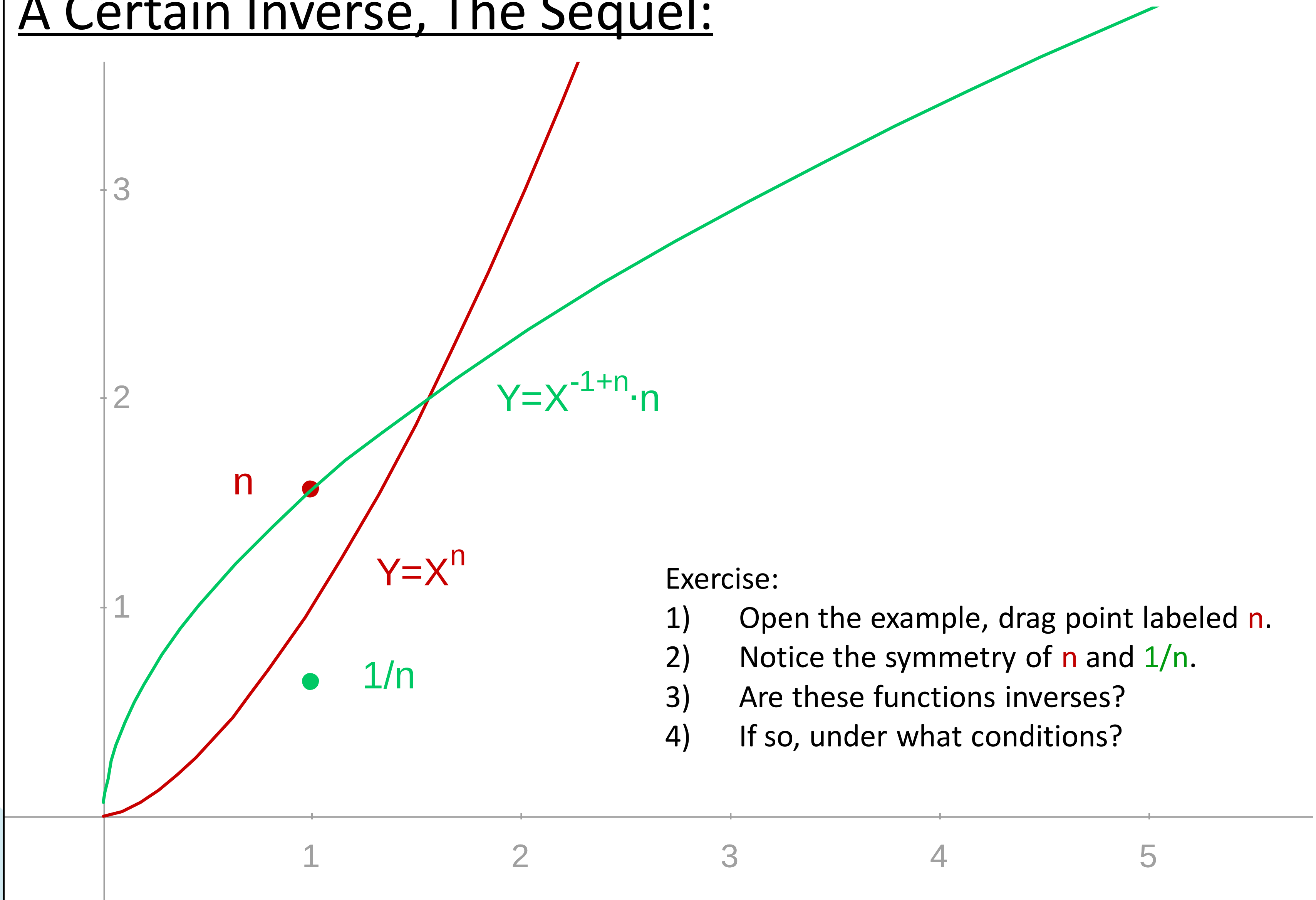
$$Undiff\left(\frac{Diff(x^3)}{3}\right) = Undiff(x^2) = \frac{x^3}{3} + C$$

$$\int x^2 dx = \frac{x^3}{3} + C$$

-Li Wei

Lecture19-ACertainInverseSequel.gx

A Certain Inverse, The Sequel:



Exercise:

1) Open the example, drag point labeled n.
2) Notice the symmetry of n and 1/n.
3) Are these functions inverses?
4) If so, under what conditions?

$$\int cf(x)dx = c\int f(x)dx$$

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

```
create_list(integrate(f(x)^n,x),n,1,5);
```

$$\left[\int f(x)\,dx,\ \int f(x)^2\,dx,\ \int f(x)^3\,dx,\ \int f(x)^4\,dx,\ \int f(x)^5\,dx\right]$$

End