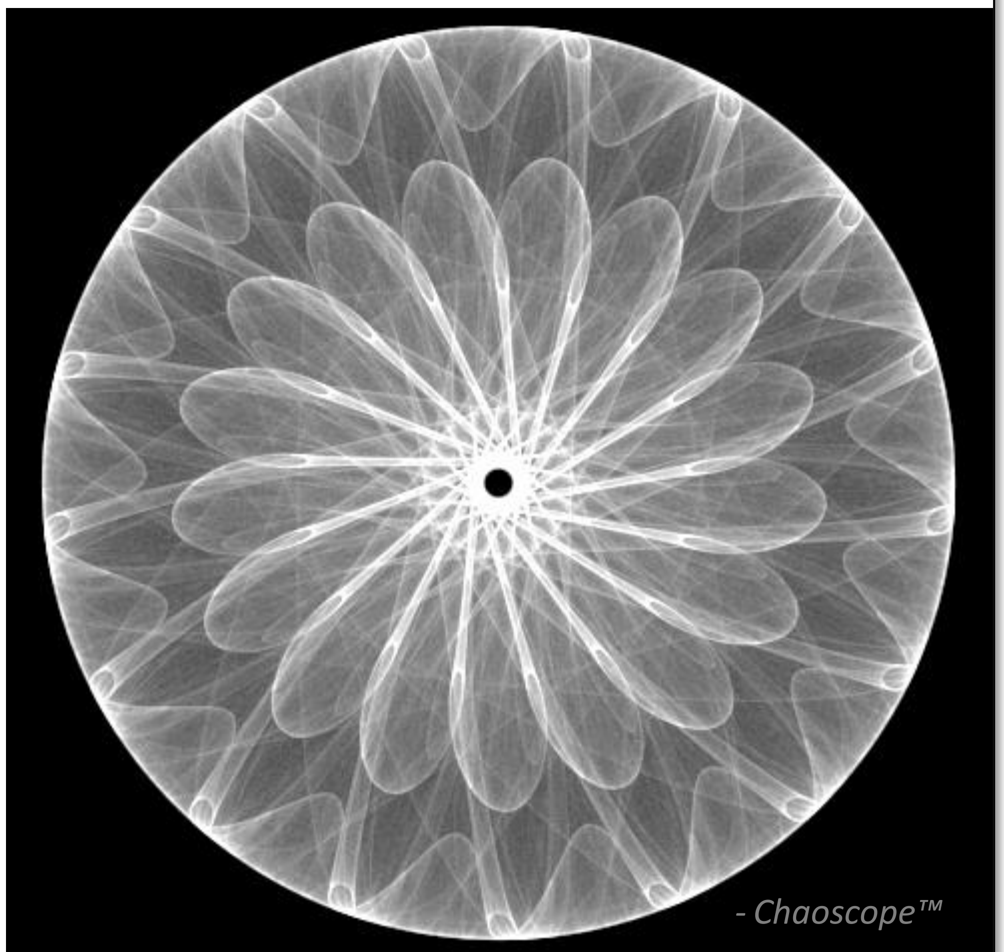
Learning Calculus With Geometry Expressions™

by L. Van Warren

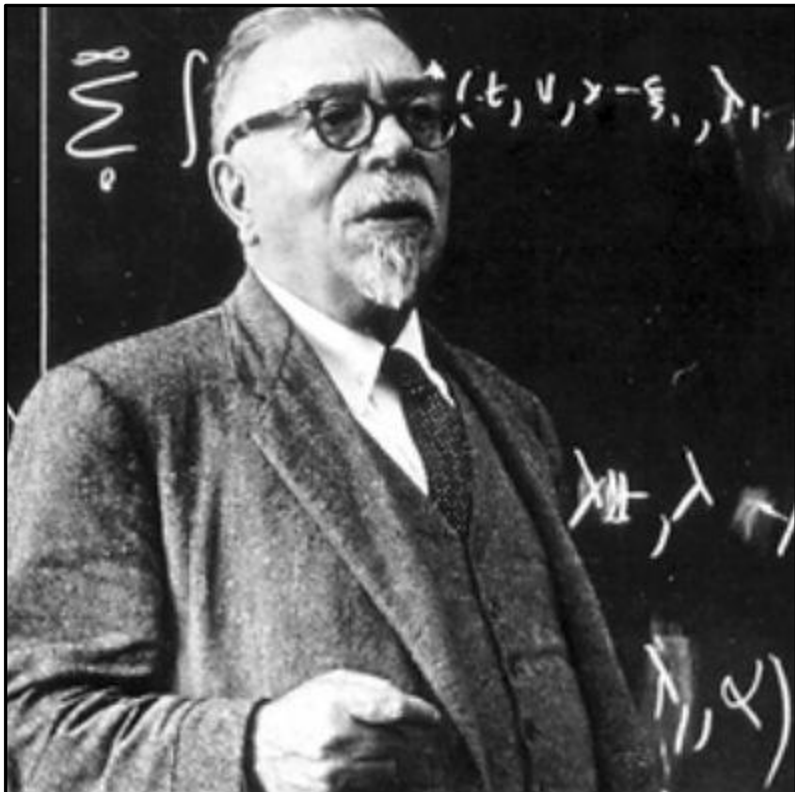
Lecture 17: Derivative Tests & Mean Value Theorem



Chapter 4: Rates and Extremes

Lecture	Topic
14	Rates of Change
15	Related Rates
16	Extrema – Maxima and Minima
17	**Derivative Tests and Mean Value Theorem**
18	Optimization

Inspiration



- *Image courtesy MIT*

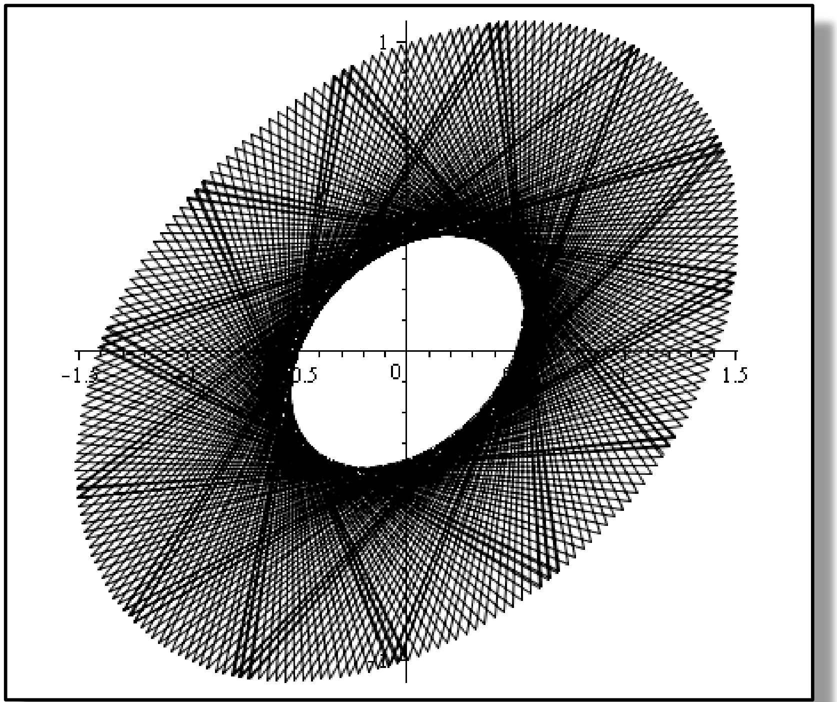
Norbert Weiner 1894 - 1964
MIT Mathematician

Founded Cybernetics
Formalized Feedback
Founded Communication Theory
(with Nyquist, Shannon & Pierce)

Studied with : Karl Schmidt
Bertrand Russell
G.W. Hardy

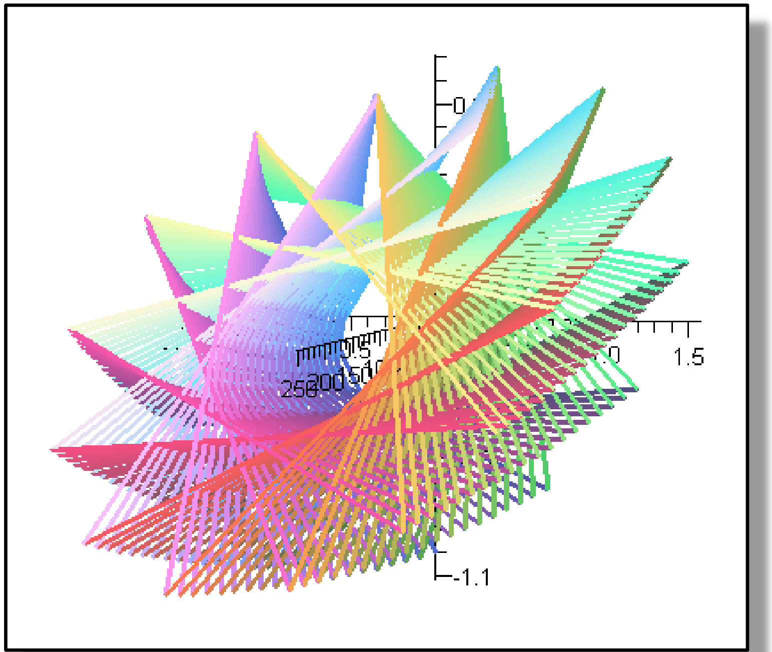
Worked on: Brownian Motion
Fourier Analysis

Iterated Equations:
Mathematical Feedback



$$\begin{bmatrix} -1.0 & 1.0 \\ -0.5 & -1.0 \end{bmatrix}$$

250 Iterations



$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_n$$

Derivative Tests and Mean Value Theorem:

In the previous lecture we learned techniques for finding the extrema of functions, the location of maximum and minimum values.

In this lecture we will extend the utility of these techniques and cover two important theorems: Rolle's Theorem and the Mean Value Theorem.

We will then use those in a practical application utilizing the sinc function.

After that we will look at concavity and other tests that yield insight into the functions whose behavior we seek to understand.

Rolle's Theorem

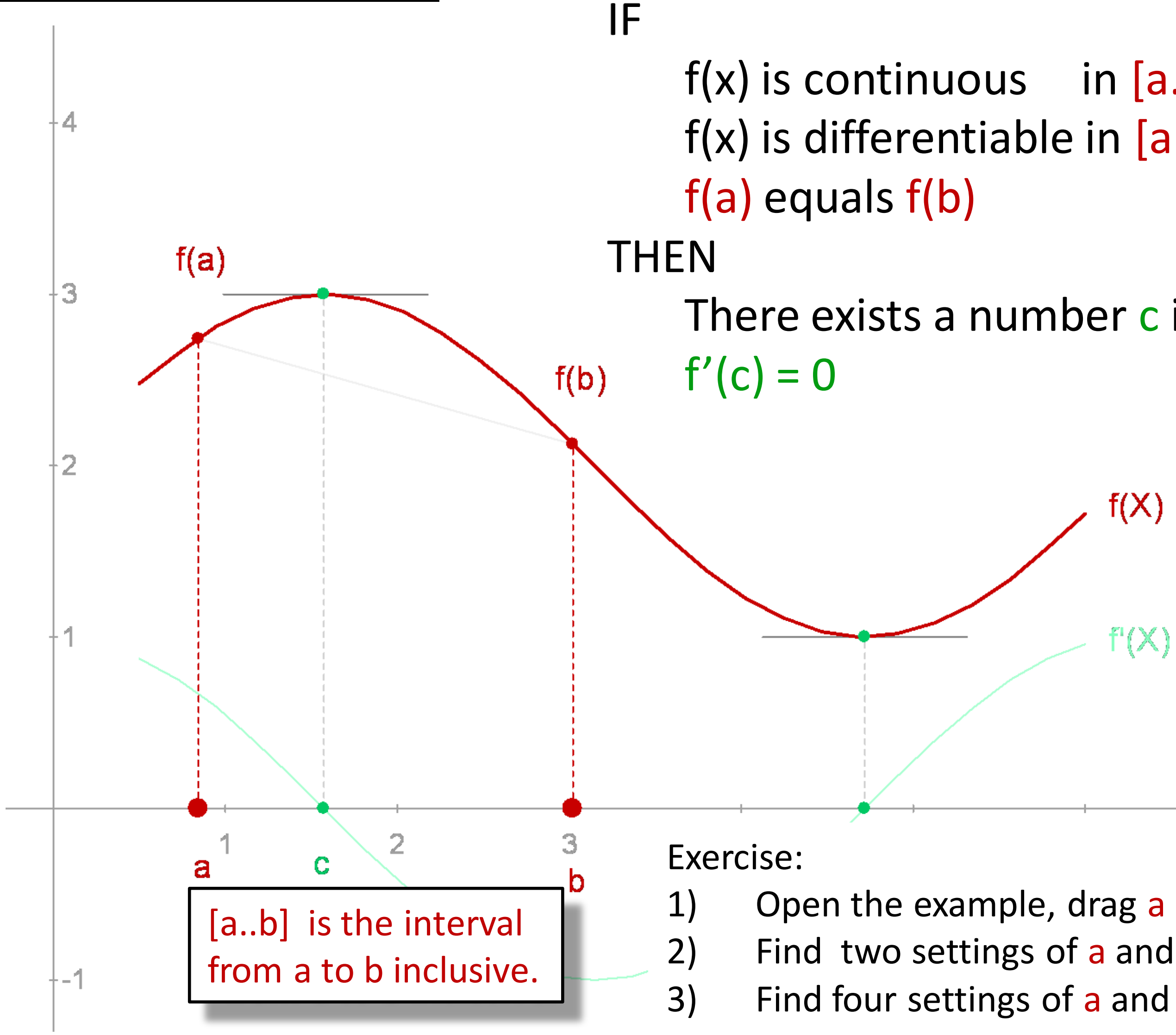
Lecture17-RollesTheoremForSin.gx

IF

f(x) is continuous in [a..b] and
f(x) is differentiable in [a..b] and
f(a) equals f(b)

THEN

There exists a number c in [a..b] such that
f'(c) = 0



Exercise:

1) Open the example, drag a and b.
2) Find two settings of a and b that satisfy Rolle's.
3) Find four settings of a and b that do not.

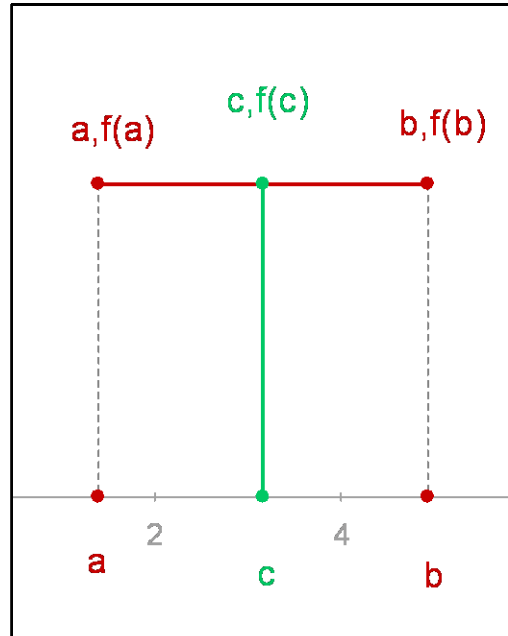
Proving Rolle's Theorem

Lecture17-ProvingRollesCase1.gx

f(x) continuous in [a..b]
f(x) differentiable in [a..b]
f(a) = f(b)

→

c exists in [a..b]
such that
f'(c) = 0



Case I:

IF

f(x) = constant

THEN

f'(x) = 0,

AND

c is any x in [a..b]

Exercise:

1) Open the example.
2) Drag all six points notice truth unchanged.

IF f'(x) = 0 for all x in [a..b]

THEN f(x) is constant on [a..b]

Lecture17-ProvingRollesCase2.gx

Proving Rolle's Theorem

f(x) continuous in [a..b]
f(x) differentiable in [a..b]
f(a) = f(b)

$\longrightarrow$

c exists in [a..b]
such that
f'(c) = 0

Case II:

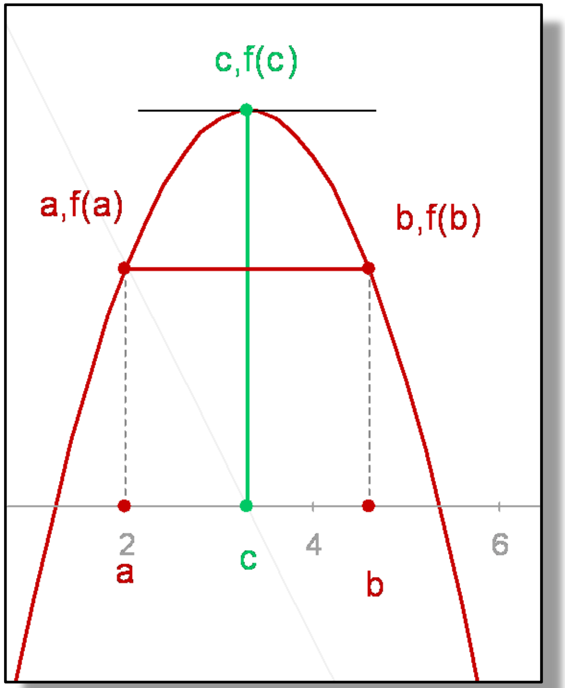
f(a) = f(b)

f(x) > f(a) for some x in [a..b]

max c is in [a..b]

By EVT f(x) has a max c is in [a..b]

Thus f(x) has a local max c and f'(c) exists and f'(c) = 0 by Fermat's Theorem



Exercise:

1) Open the example.
2) Drag all five points notice truth is unchanged.

Proving Rolle's Theorem

Lecture17-ProvingRollesCase3.gx

f(x) continuous in [a..b]
f(x) differentiable in [a..b]
f(a) = f(b)

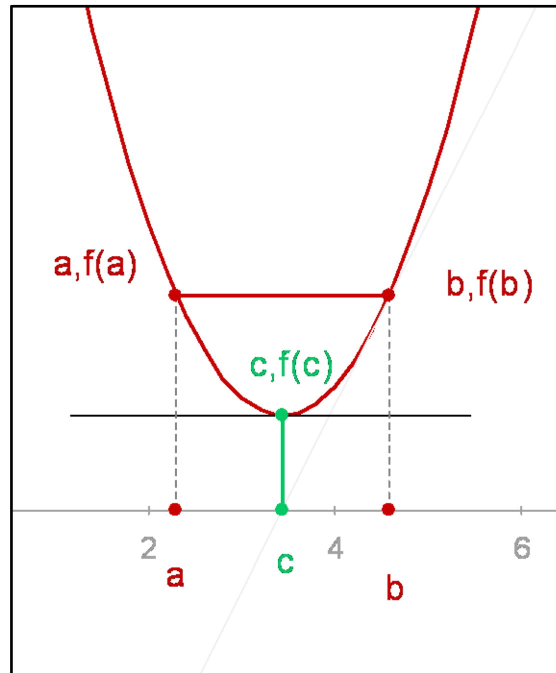
$\longrightarrow$

c exists in [a..b]
such that
f'(c) = 0

Case III:

f(a) = f(b)
f(x) < f(a) for some x in [a..b]
min c is in [a..b]
By EVT f(x) has a min c is in [a..b]

Thus f(x) has a local min c and f'(c) exists and f'(c) = 0 by Fermat's Theorem



Exercise:

1) Construct this case using Lecture17-ProvingRollesCase2.gx and Lecture17-ProvingRollesCase2.wxm

Proving Rolles Case 2: wxMaxima™

```
f(X):=k+X*p-(X-h)^2;
```

$$f(X):=k+X\,p-(X-h)^2$$

```
f(a);
```

$$a\,p+k-(a-h)^2$$

```
f(b);
```

$$b\,p+k-(b-h)^2$$

```
solve(f(a)=f(b),b);
```

$$[\,b=p+2\,h-a\,,\ b=a\,]$$

```
f(second(first(%)));
```

$$p(p+2\,h-a)-(p+h-a)^2+k$$

1) Write the equation.
2) Find f(a).
3) Find f(b).
4) Find b such that f(a) = f(b).
5) Back substitute to find f(b).

Proving Rolles Case 2: wxMaxima™

```
fp(X):=diff(f(X),X);
```

$$fp(X):=diff(f(X),X)$$

```
solve(fp(X),X);
```

$$[X=\frac{p+2h}{2}]$$

```
f(second(first(%)));
```

$$-\left(\frac{p+2h}{2}-h\right)^2+\frac{p(p+2h)}{2}+k$$

```
fp(X);
```

$$p-2(X-h)$$

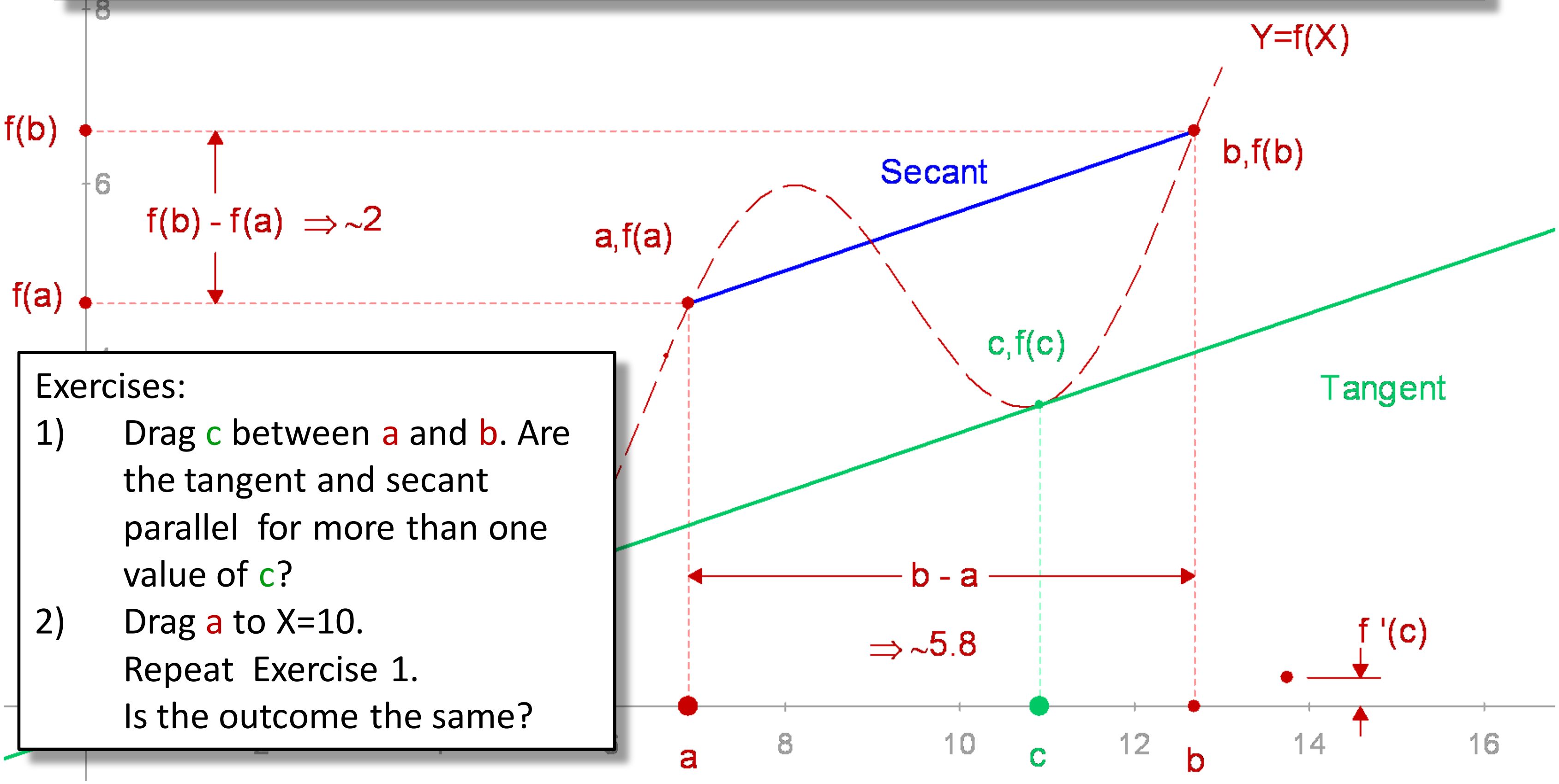
1) Differentiate f(X).
2) Solve for c, the critical X.
3) Find f(c), the critical Y.
4) Find f'(X) for drawing.

The Mean Value Theorem: MVT

Lecture17-MeanValueTheorem1.gx

IF $f(x)$ is continuous in [a..b] and $f(x)$ is differentiable in [a..b]

THEN There exists a number c in [a..b] such that: $$f'(c) = \frac{f(b) - f(a)}{b - a}$$



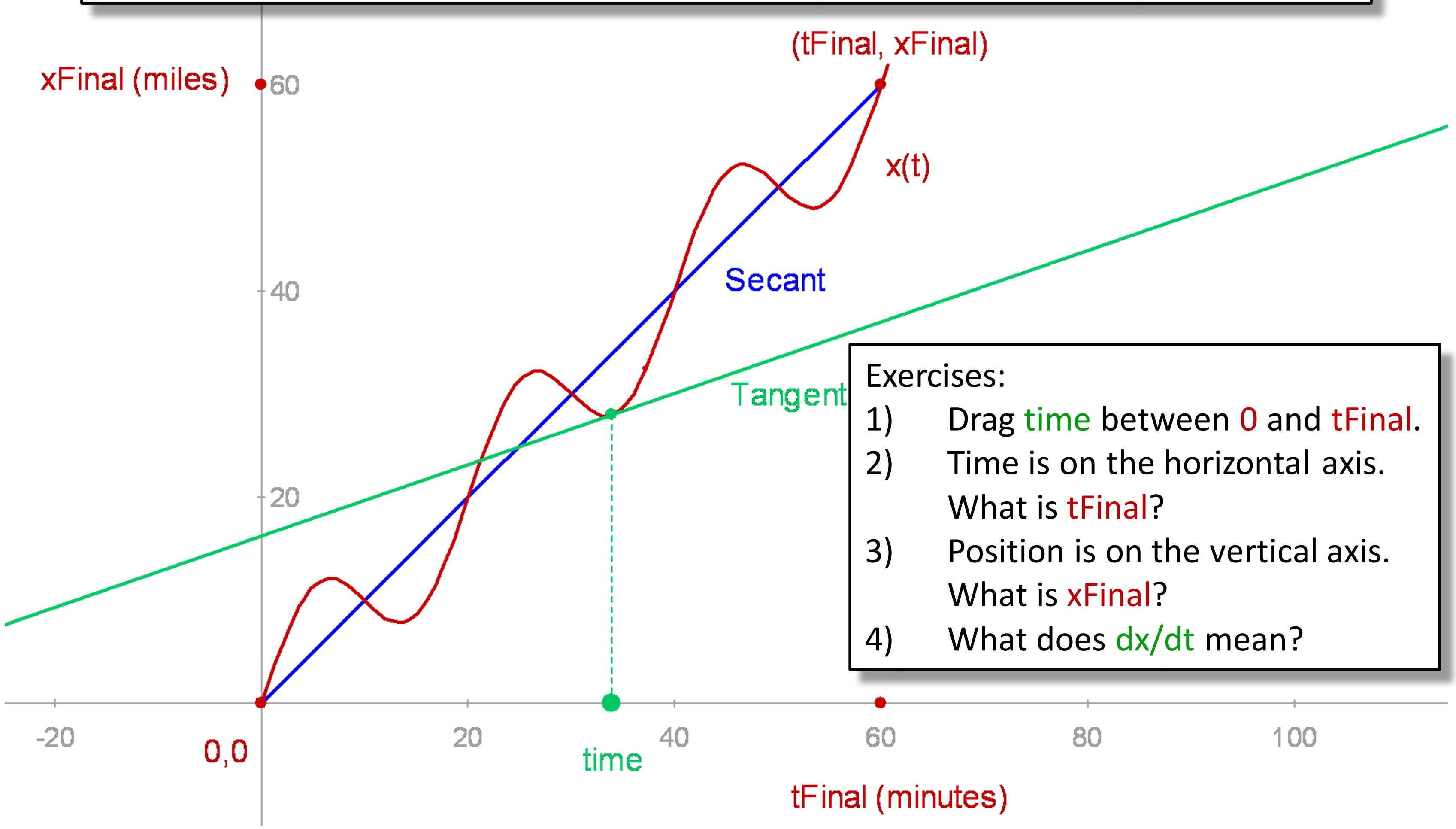
Exercises:

1) Drag c between a and b. Are the tangent and secant parallel for more than one value of c?
2) Drag a to X=10. Repeat Exercise 1. Is the outcome the same?

MVT: Practical Example

Lecture17-MeanValueTheorem2.gx

If a car travels 60 miles in 60 minutes, averaging 1 mile per minute, then at some time its instantaneous speed was 1 mile per minute.



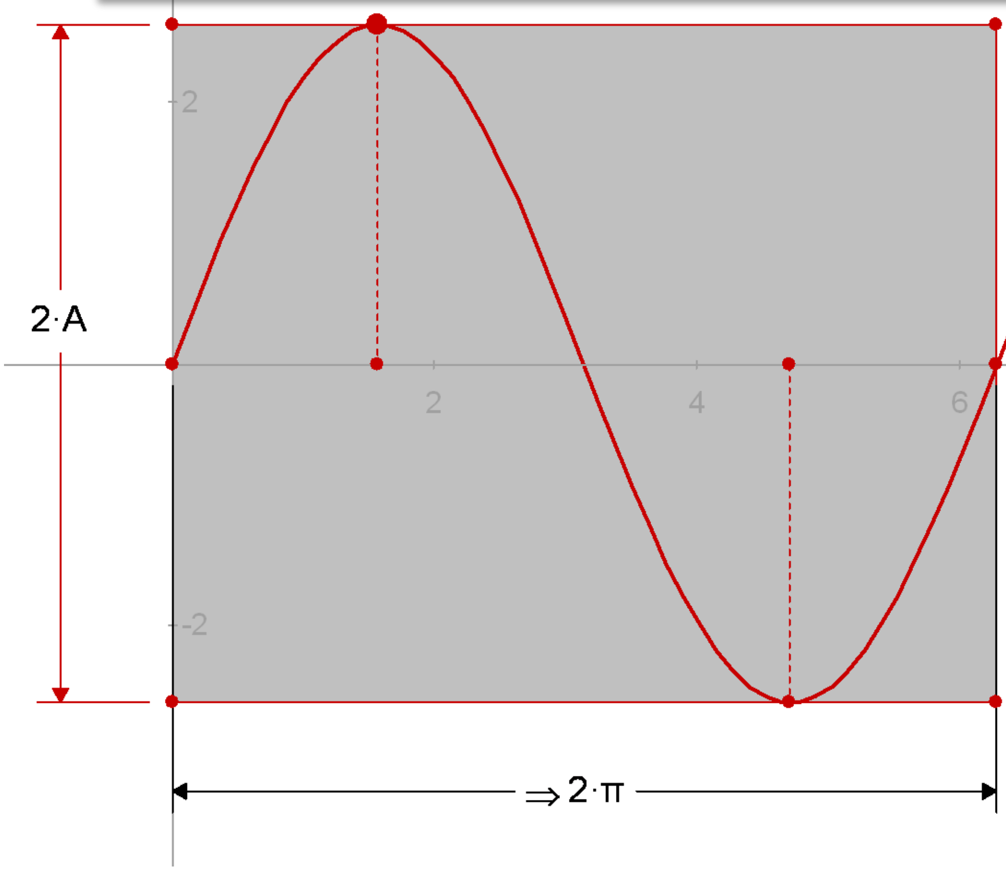
Exercises:

1) Drag time between 0 and tFinal.
2) Time is on the horizontal axis. What is tFinal?
3) Position is on the vertical axis. What is xFinal?
4) What does dx/dt mean?

A Short Aside: Boxed Sine Wave

Lecture17-BoxedPiSin.gx

Sine and cosine are both periodic over an interval of width 2π.
They make one complete cycle in 2π radians.
Notice that sin(x) is not symmetric about the vertical axis.



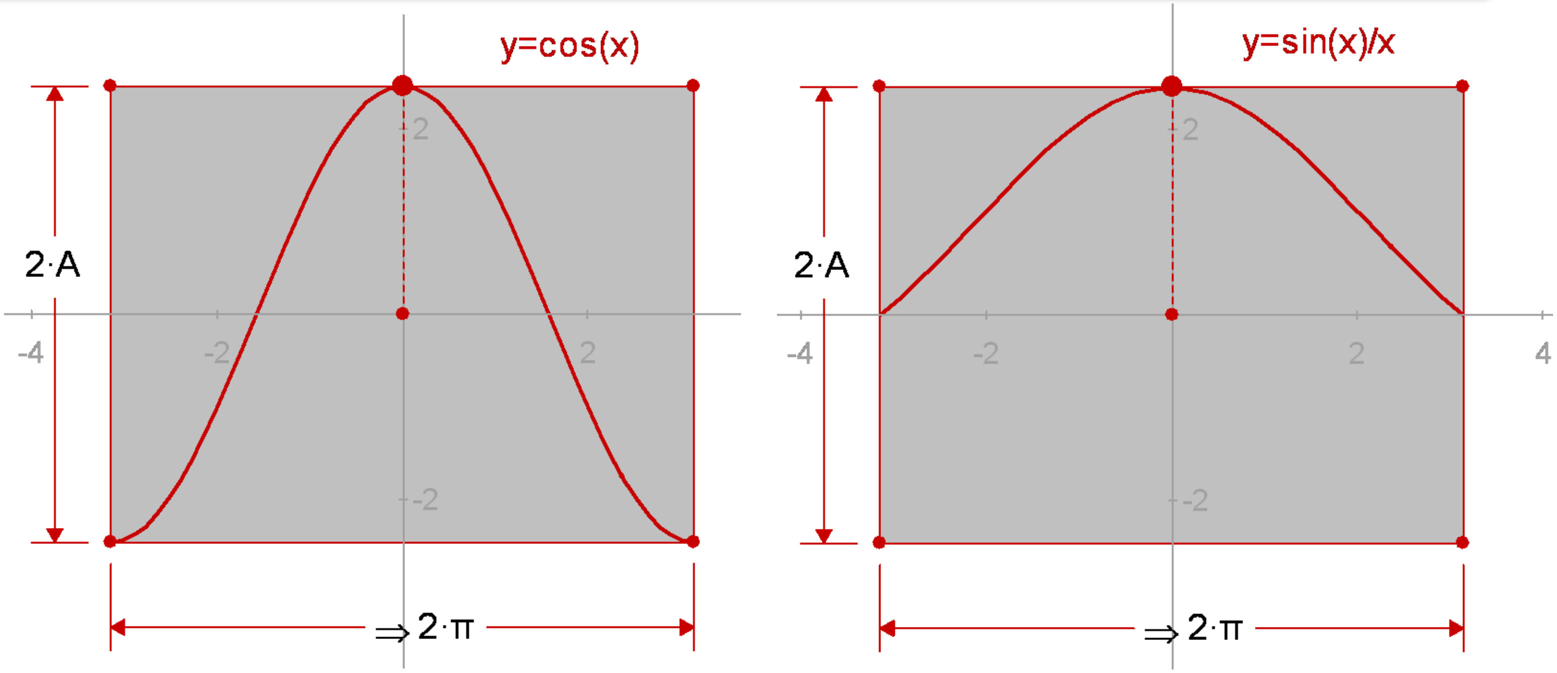
Exercises:

1) Drag A.
2) What is the width of a cycle?
3) What is the height of a cycle?
4) For what value of Amplitude is the bounding region of a sine wave cycle square?

Boxed Cosine Wave and Sinc

Lecture17-BoxedPiCos.gx
Lecture17-BoxedPiSinc.gx

Notice the symmetry of cos(x) and sin(x)/x about the vertical axis.
How did a function containing sin(x) suddenly become symmetric?

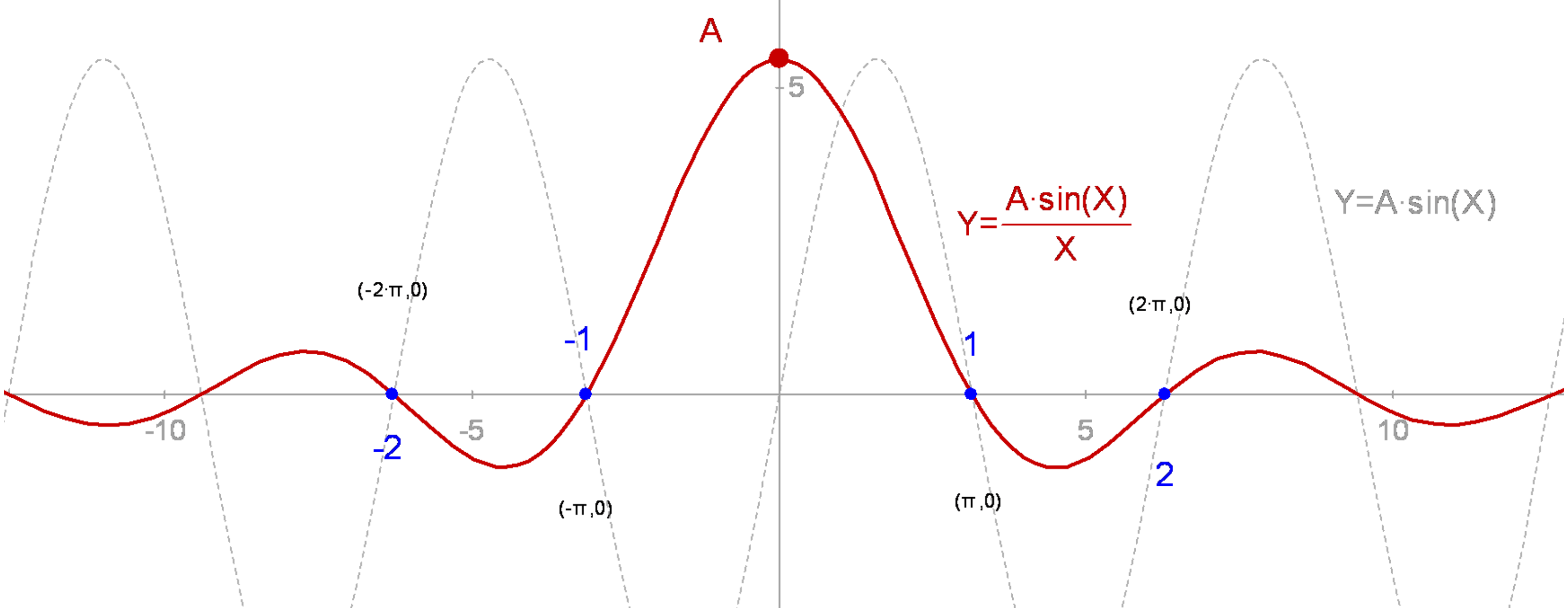


Exercises: 1) Notice we did not get a full "cycle" with the sinc function.
2) Compute the width of the first cycle of sinc(x) = sin(x)/x.

Sinc Roots

Lecture17-SincCycleWidth.gx

We can find the zeroes of the sinc cycle by finding its roots.
These are multiples of $n \cdot \pi$, coincident with the roots of sin(X).

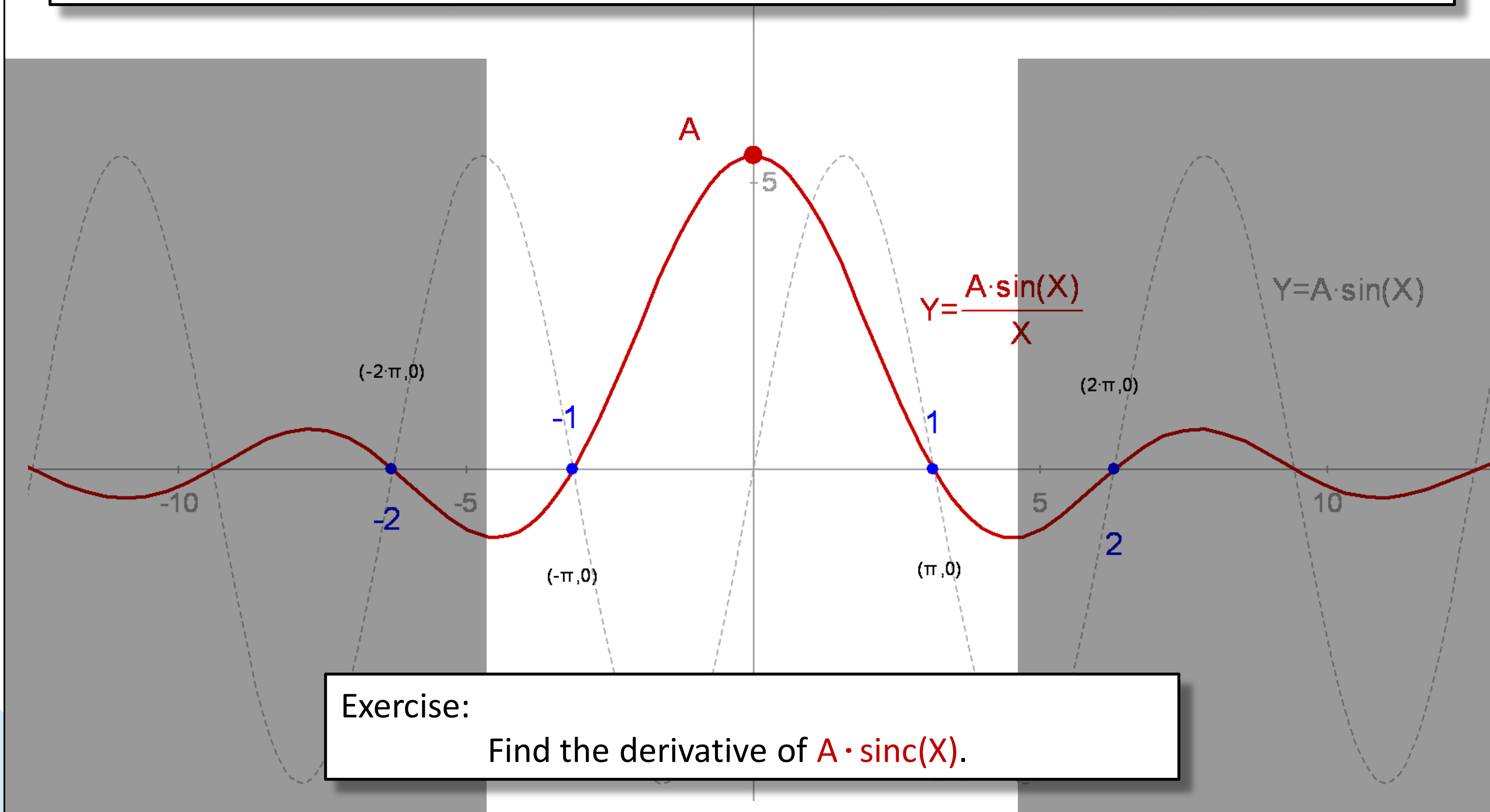


Exercises: 1) Drag A up and down.
2) Does the value of A affect the roots of sinc(X)?

Sinc Cycle Width

Lecture17-SincCycleWidth.gx

Finding the width of the central sinc cycle requires the critical points.
These are not multiples of $n\cdot\pi$, nor are they zeroes of sinc(X).



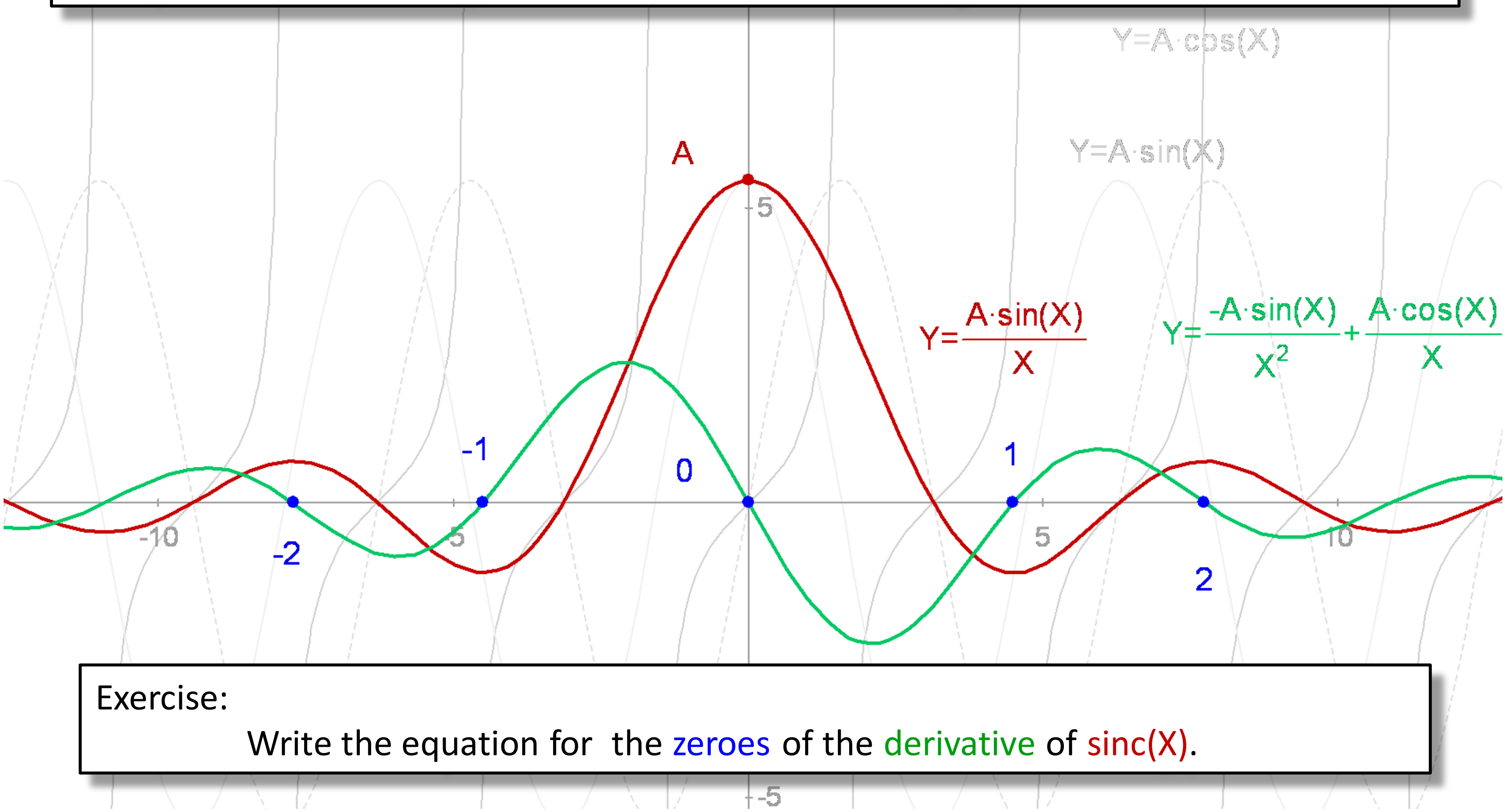
Exercise:

Find the derivative of $A\cdot\mathrm{sinc}(X)$.

Sinc Cycle Width

Lecture17-SincCriticalPoints.gx

We can find the sinc cycle width by finding the roots named -1 and 1. They are obtained by numerical solution using Newtons' Method.



Exercise:

Write the equation for the zeroes of the derivative of sinc(X).

Sinc Cycle Width

Lecture17-SincCriticalPoints.wxm

```
Y(X):=A*sin(X)/X;
```

$$Y(X):=\frac{A\sin(X)}{X}$$

```
diff(Y(X),X);
```

$$\frac{A\cos(X)}{X}-\frac{A\sin(X)}{X^2}$$

```
solve(%,X);
```

$$[X=\frac{\sin(X)}{\cos(X)}]$$

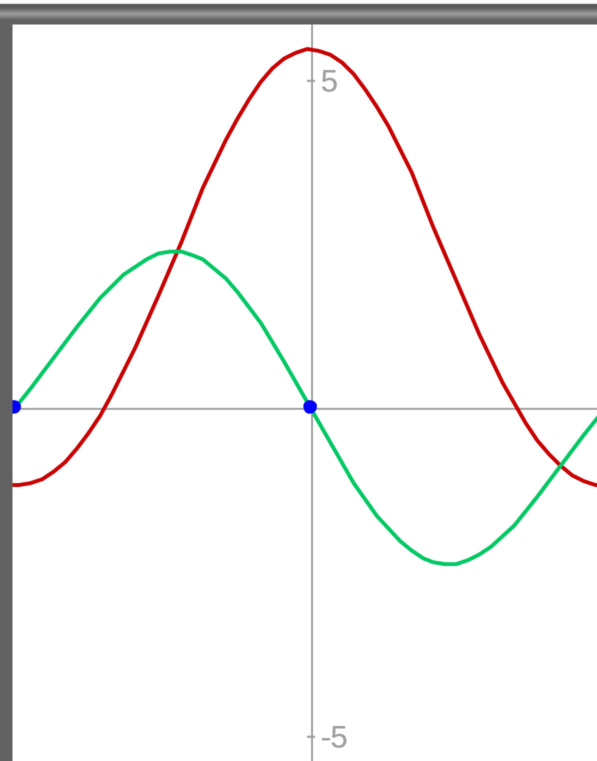
Finding Critical Points does not always have a closed-form solution. Such cases require numerical techniques such as Newtons' Method shown here.

```
load(newton1)$

newton(tan(x)-x, x,4.5,1/10^6);
4.493409457909247

newton(tan(x)-x, x,7.7,1/10^6);
7.725251836938464
```

Exercise: Where does the `tan(x)` term come from?

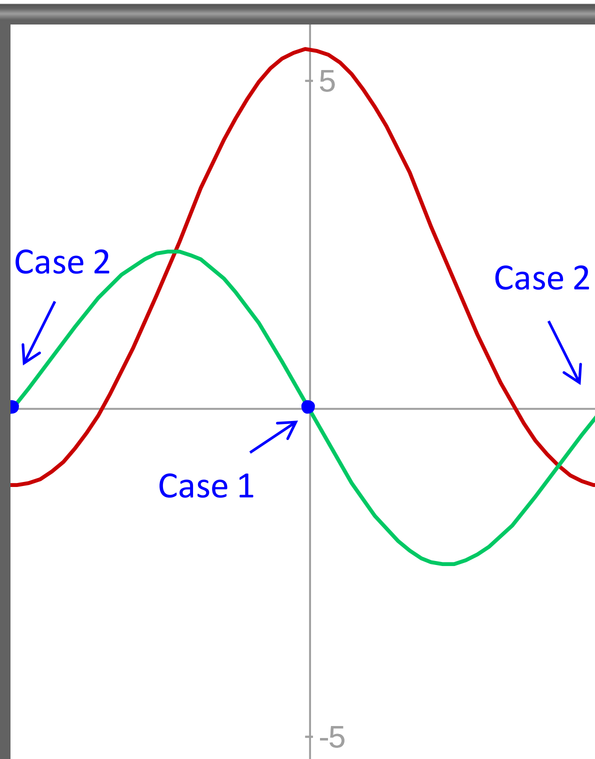
Increasing/Decreasing "ID" Test

1) If $f'(x) > 0$ on [a..b] then $f(x)$ is increasing on [a..b]
2) If $f'(x) < 0$ on [a..b] then $f(x)$ is decreasing on [a..b]



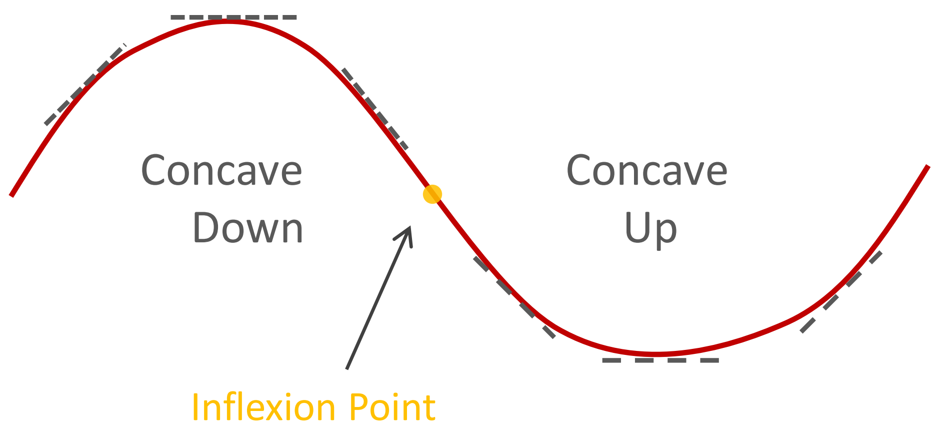
The First Derivative Test

1) If f’(x) changes + to - at c, then f(x) has a local max at c
2) If f’(x) changes - to + at c, then f(x) has a local min at c
3) If f’(x) does not flip sign at c, then f(x) has NO max/min at c



Concavity Upward and Downward

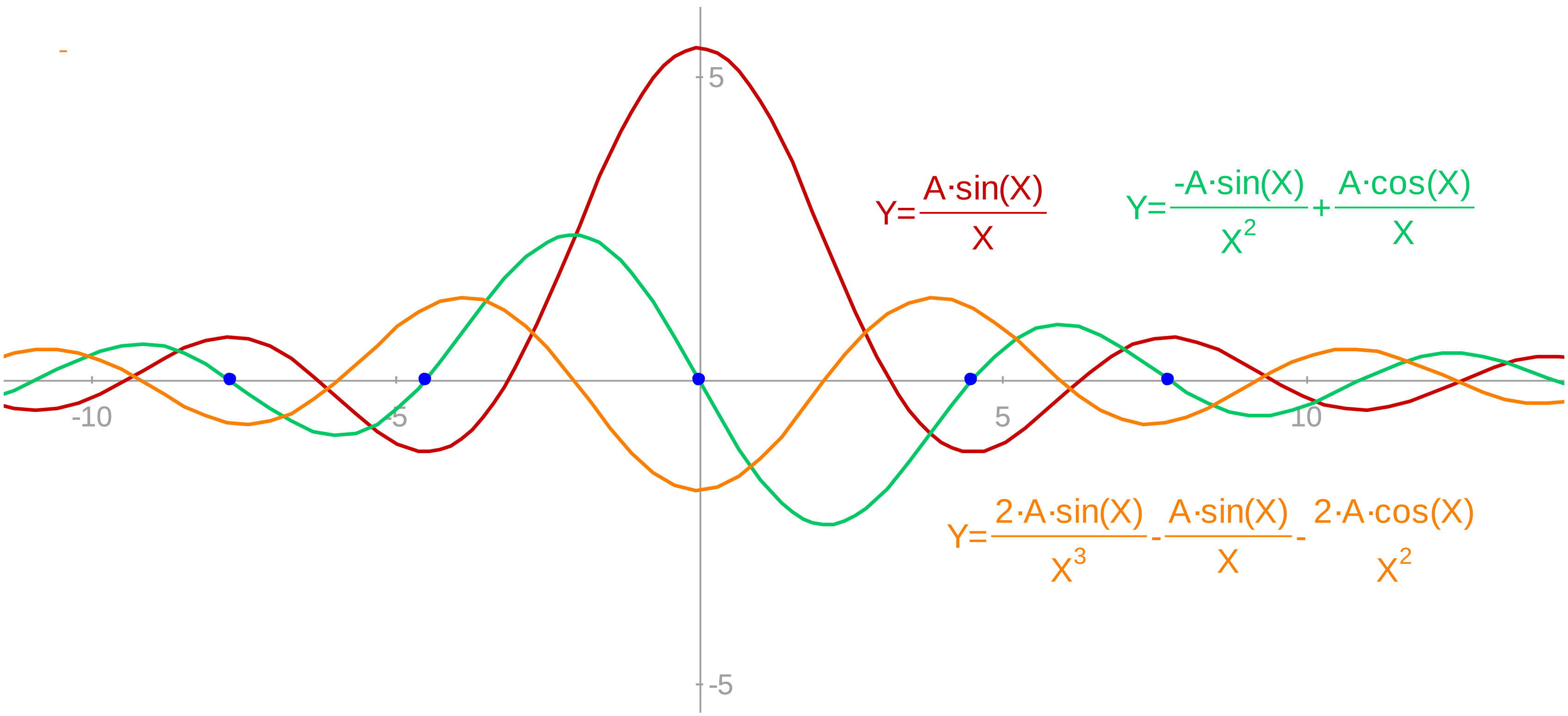
1) If f(x) lies above its tangents on [a, b] it is concave upward.
2) If f(x) lies below its tangents on [a, b] it is concave downward.



Concavity Test

Lecture17-SincSecondDeriv.gx

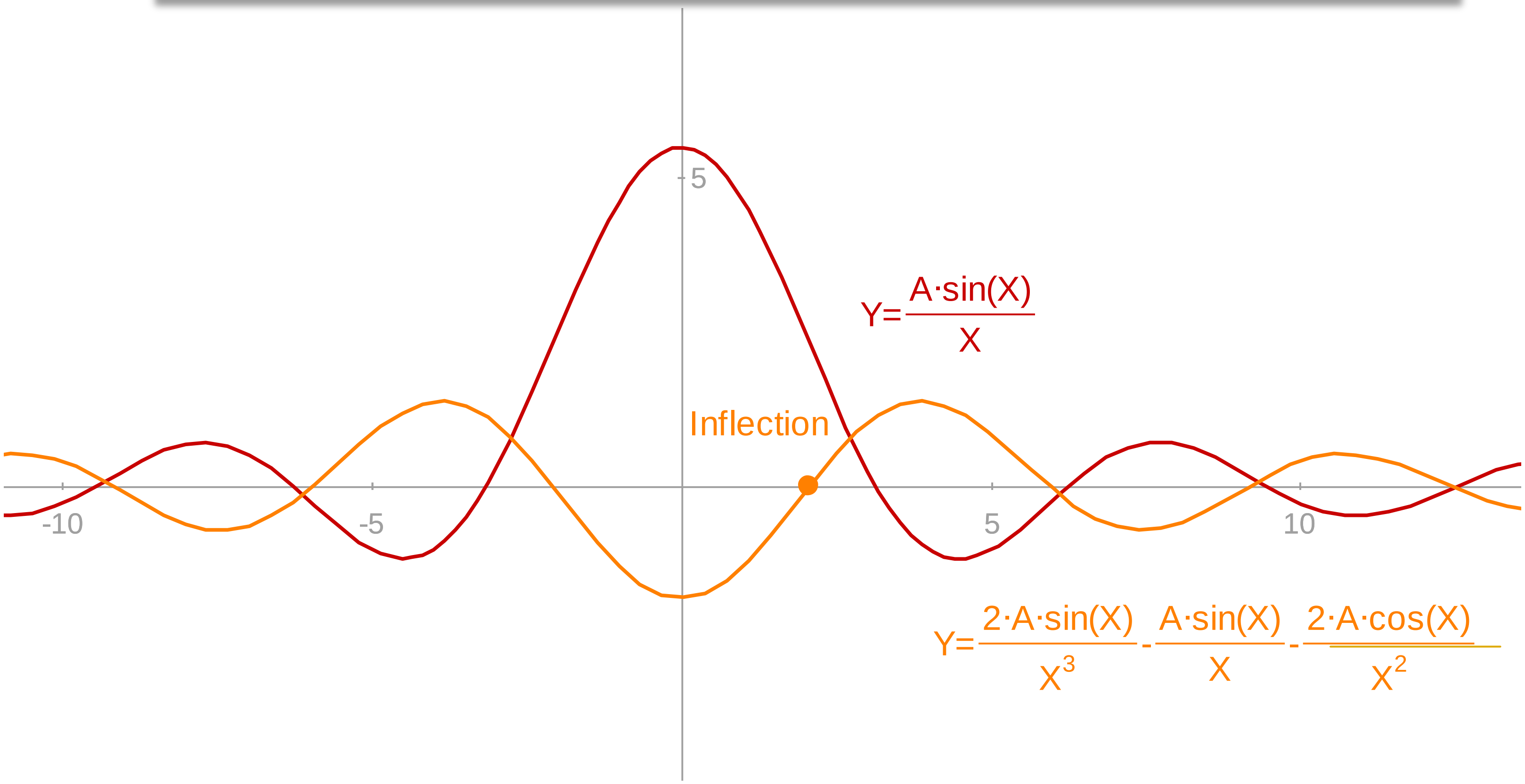
1) If $f''(x) > 0$ for on [a..b] then f(x) is concave upward on [a..b]
2) If $f''(x) < 0$ for on [a..b] then f(x) is concave down on [a..b]



Inflection Point

Lecture17-SincSecondDeriv.gx

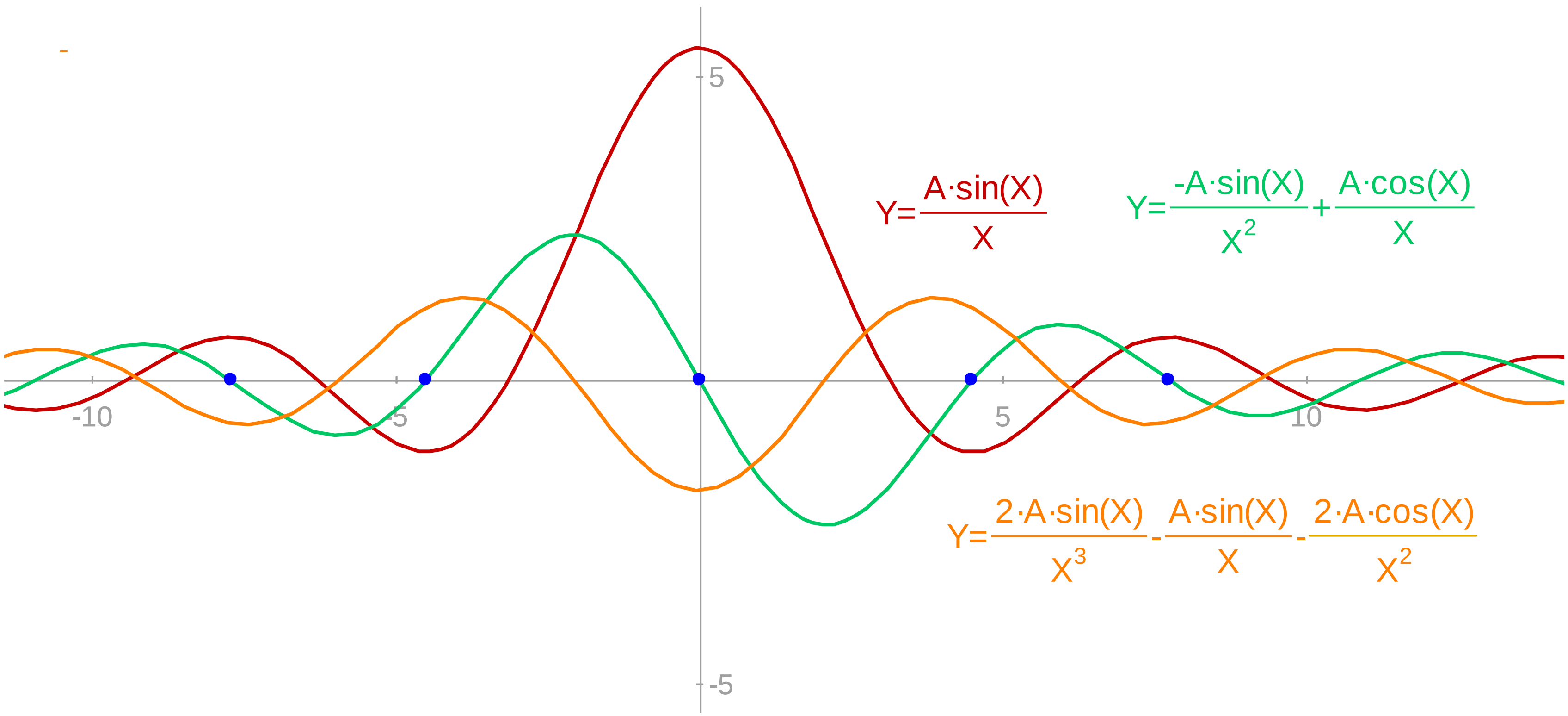
f''(x) changes sign as concavity changes from up to down.
This is called an inflection point.

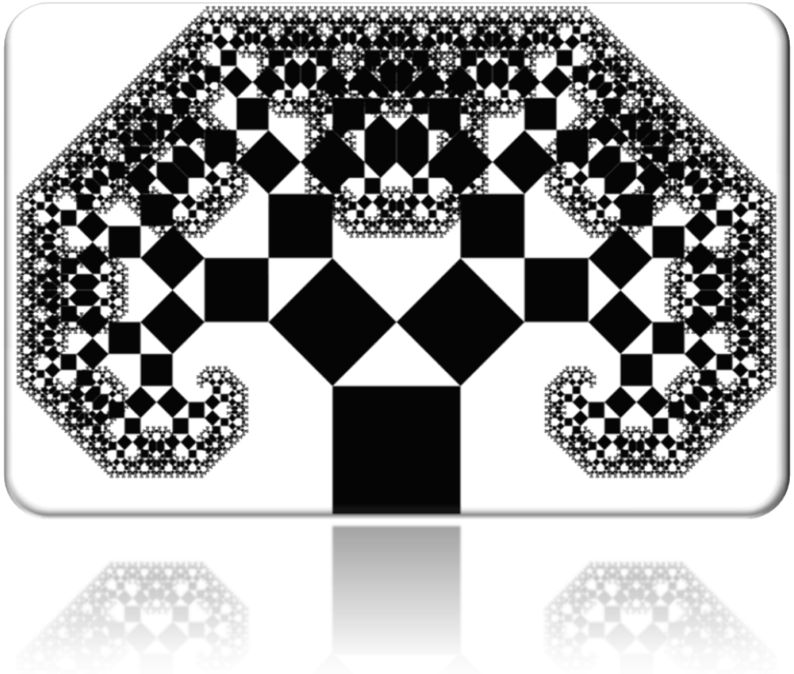


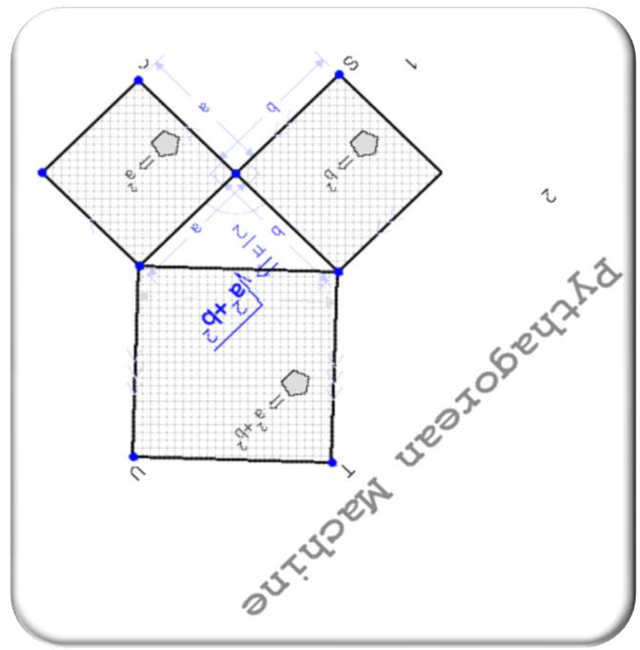
Second Derivative Test

Lecture17-SincSecondDeriv.gx

1) If f'(c) = 0 and f''(c) > 0, then f(x) has a minimum at c.
2) If f'(c) = 0 and f''(c) < 0, then f(x) has a maximum at c.







End