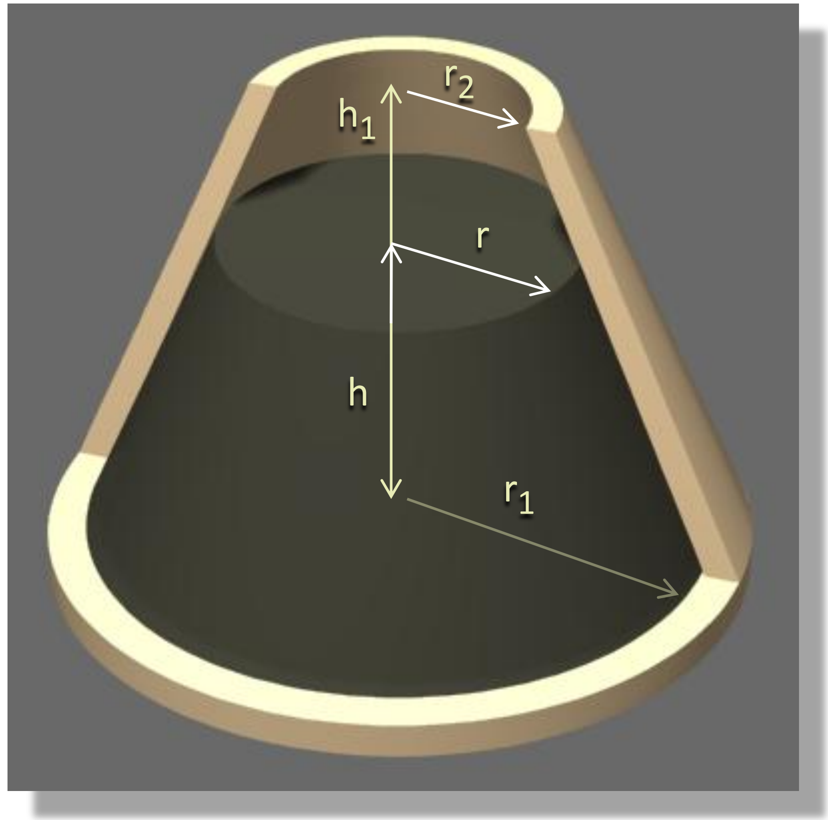
Learning Calculus With Geometry Expressions™

by L. Van Warren

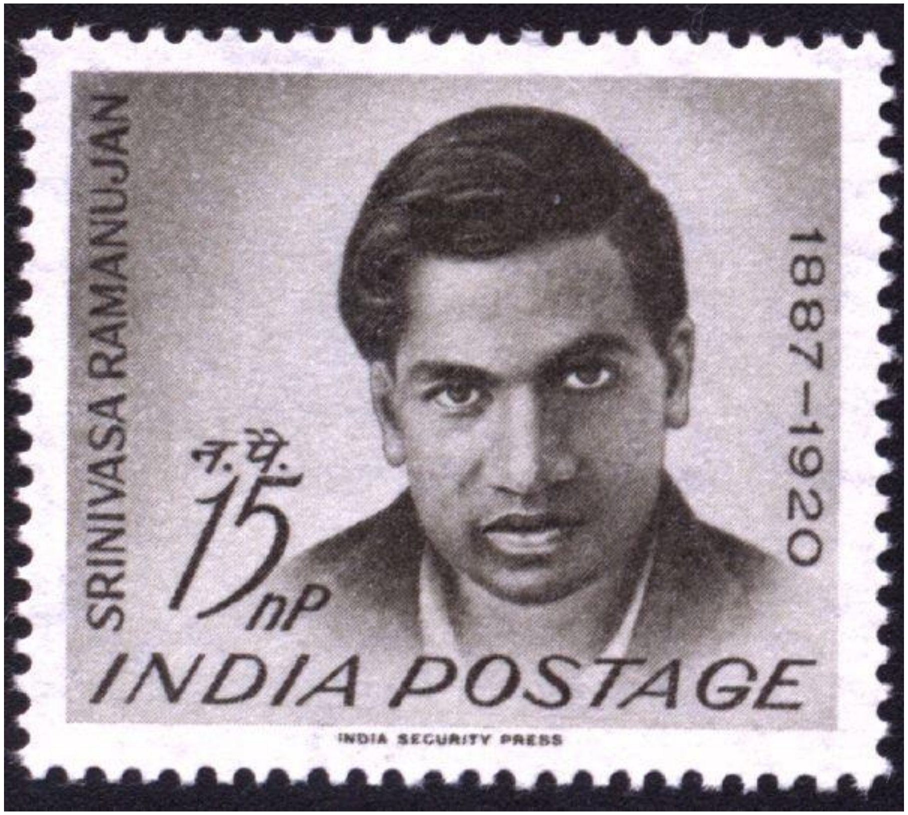


Lecture 15: Related Rates

Chapter 4: Rates and Extremes

LECTURE	TOPIC
14	RATES OF CHANGE
15	**RELATED RATES**
16	EXTREMA – MAXIMA AND MINIMA
17	DERIVATIVE TESTS AND MEAN VALUE THEOREM
18	OPTIMIZATION

Inspiration



Srinivasa Ramanujan

Borrowed [Carr's Synopsis]() at age 15.

Worked In Poverty and Illness

- Number Theory
- Infinite Series
- Continued Fractions

Passed Away At 32

- Left Notebooks w > 3900 Entries
- Most Proofs Erased
  Because Paper Was Too Expensive

Ramanujan Inspired 4417 Theorems

Dougall-Ramanujan identity
Hardy-Ramanujan number
Landau-Ramanujan constant
Ramanujan's congruences
Ramanujan-Nagell equation
Ramanujan-Peterssen conjecture
Ramanujan-Skolem's theorem
Ramanujan-Soldner constant
Ramanujan summation
Ramanujan theta function

Ramanujan graph
Ramanujan's tau function
Ramanujan Mathematical Society
Ramanujan prime
Ramanujan's constant
Ramanujan's lost notebook
Ramanujan's sum
Rogers-Ramanujan identity
SASTRA Ramanujan Prize
Srinivasa Ramanujan Centre
Srinivasa Ramanujan Medal
The Ramanujan Journal

Related Rates

When we take the derivative, we discover rate relationships. Expressions like ***dy/dx***, ***dx/dt***, or ***ds/dx*** are all expressions of how fast one variable changes with respect to another.

For compactness we sometimes substitute a single symbol for the derivative, as in:

$$v = dx/dt$$

This substitution hides information and provides leverage, one symbol taking the place of five.

Related Rates

One rate $\boldsymbol{a}$, can be related to another $\boldsymbol{v}$, as in:

$$a = dv/dt$$

Where the rate called $\boldsymbol{a}$ is related to the rate called $\boldsymbol{v}$ by a differentiation. Two rates can also be related some other way as in:

$$dy/dx = k \cdot du/dx$$

which says, "***y*** *changes with* ***x*** like ***u*** *changes with* ***x***, times ***k***.

"Relating rates generates differential equations".

Related Rates: Cylinders

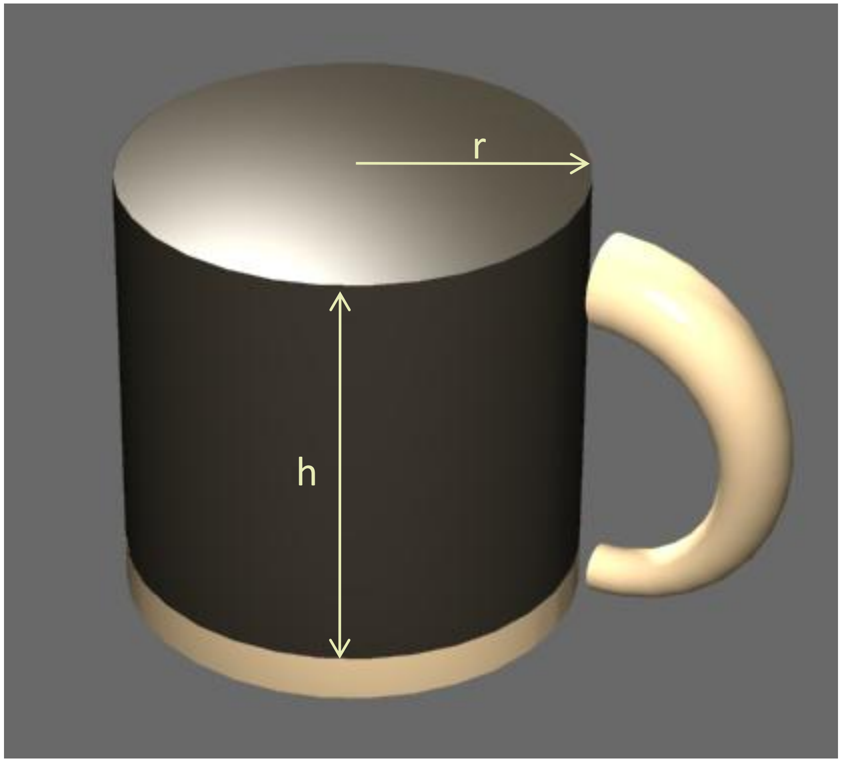
Consider filling a cup of coffee…

How is the change in fluid ***volume*** with time related to the change in fluid ***height*** with time?

Related Rates: Cylinders

We start with the expression that relates fluid volume ***V*** and fluid height ***h***:

$$V_{coffee} = Area \cdot Height$$
$$= \pi r^2 \cdot h$$



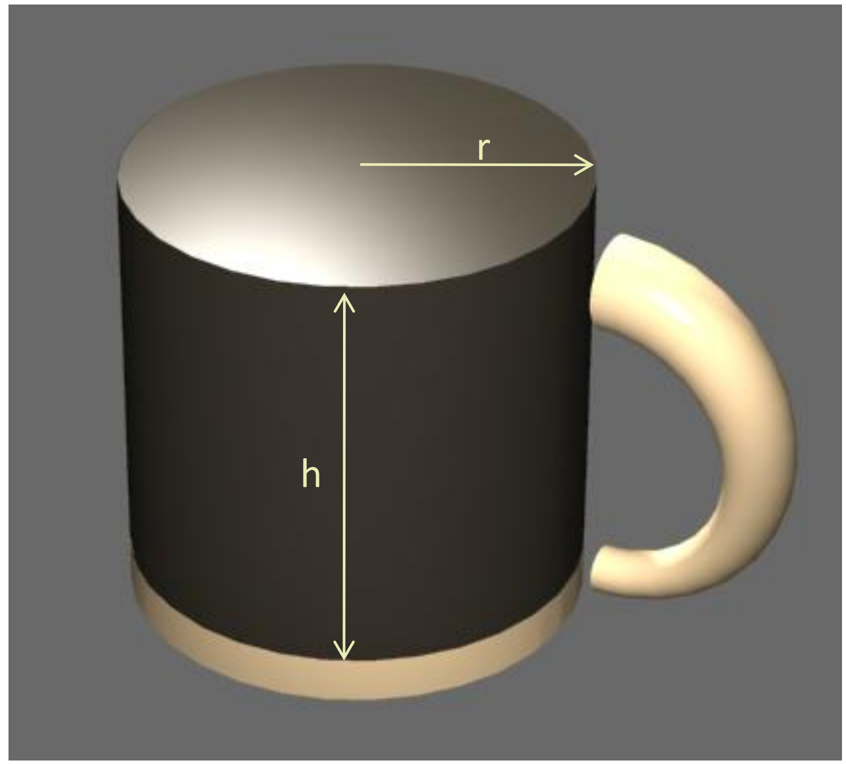
In this case, the volume of coffee is modeled as the volume of a cylinder.

We will neglect the handle!

Related Rates: Cylinders

Now we just differentiate the volume expression with respect to time:

$$\frac{dV}{dt} = \pi r^2 \cdot \frac{dh}{dt}$$



Since the radius ***r*** of the cup does not change as it fills, it is treated as a ***constant***.

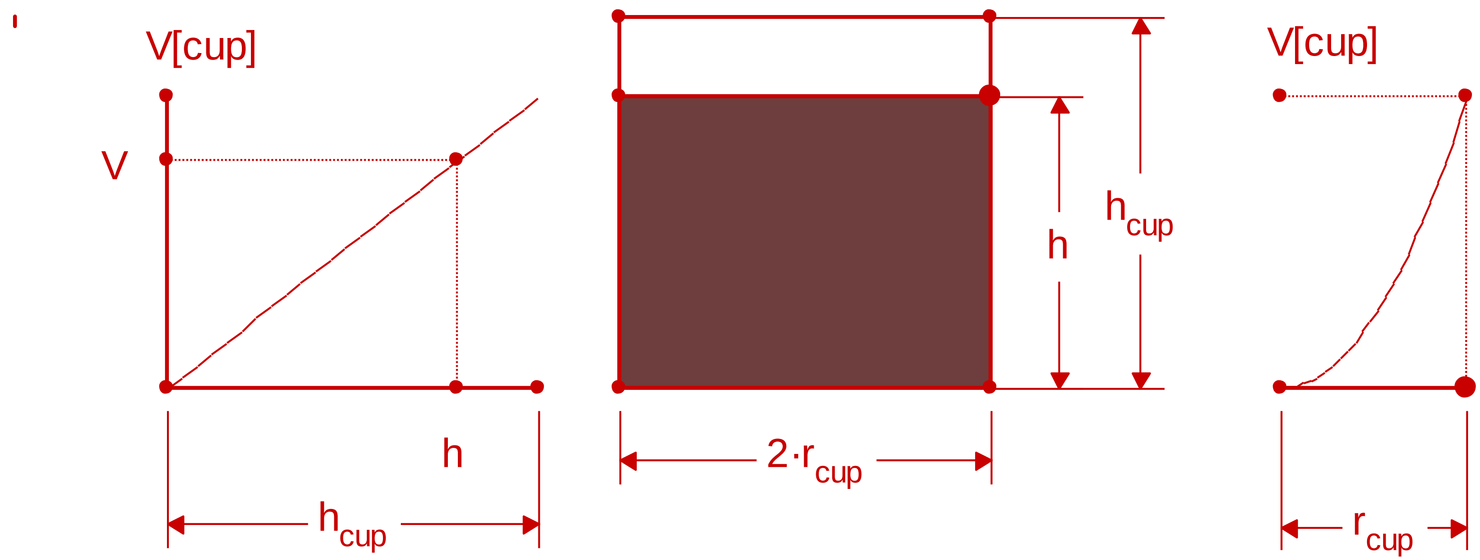
Now we have two rates related by a constant .

Filling the Cylinder

Lecture15-FillingTheCylinder.gx

EXERCISES:

1) Drag the height control. Does volume change linearly with height?
2) Animate filling the cup using the animation controls to increase h.
3) Unlock the radius in the Variables dialog. How does volume vary with radius?
4) Which increases volume more quickly, height increase or radius increase?
5) Find dh/dt as a function of dV/dt.



Filling the Cup: Radius vs. Height

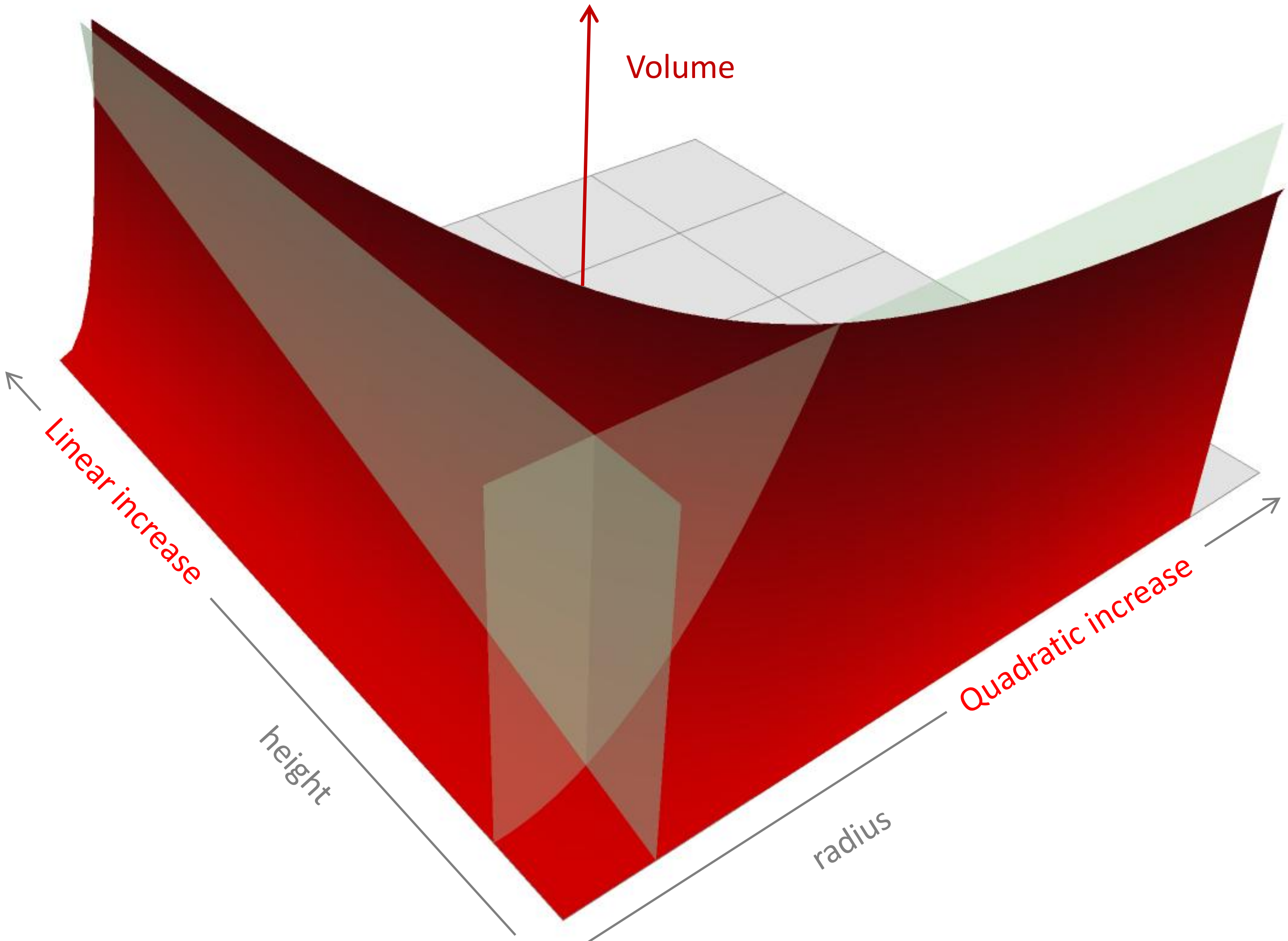
We can graph the original volume relationship for the coffee cup problem and observe that when we increase the height of the cup, the volume increases linearly. When we increase the radius of the cup, the volume increases quadratically.

The graph of the original volume relationship is a surface whose altitude (volume) is a function of two parameters, the fluid height in the cup (h) and the radius of the cup (r).

If we cut this surface at a value of constant radius we see that changes in the height of the fluid cause linear changes in volume.

If we cut this surface at a value of constant fluid level we see that changing the radius of the cup causes quadratic changes in volume.

Filling the Cylinder



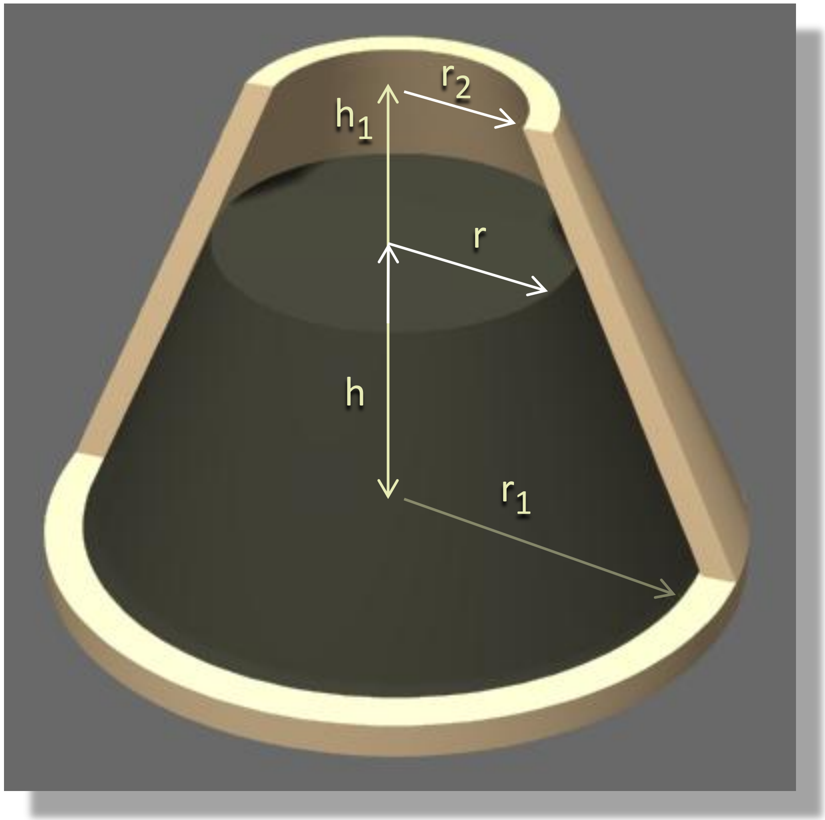
Related Rates: Cones

Consider the liquid level problem, this time for a cone…

How does
change in fluid volume with time relate to
change in fluid height   with time?

Related Rates: Cones

As before, we use the expression that relates fluid volume ***V*** and fluid height ***h***:



$$V_{coffee} = \frac{1}{3}\pi h(r_1^2 + r^2 + r_1 r)$$

The radius changes from r_1 at the container base to r_2 at the container top of height h_1.

The radius r floats at the fluid surface at a height of h.

$$r = mh + b$$

The radius versus height relation is: $$r = \frac{(r_2 - r_1)}{h_1}h + r_1$$

Related Rates: Cones

We need to eliminate the variable ***r*** for radius. Substituting the relationship of radius versus height yields liquid volume vs. h:

$$V_{coffee} = \frac{\pi h \left(\left(\frac{(r_2 - r_1)h}{h_1} + r_1 \right)^2 + r_1 \left(\frac{(r_2 - r_1)h}{h_1} + r_1 \right) + r_1^2 \right)}{3}$$

Differentiating yields:

$$\frac{dV}{dt} = \frac{\pi (r_2 h - r_1 h + h_1 r_1)^2}{h_1^2} \frac{dh}{dt}$$

Exercise: Show that this is correct by hand, then check your work with wxMaxima™.

Related Rates: Cones

Note that dV/dt changes with dh/dt under control of the term:

$$\frac{\pi(r_2h - r_1h + h_1r_1)^2}{h_1^2}$$

This term shows dV/dt per unit of change in dh/dt. We can solve for this term to find dV/dh, the change in volume with fluid height.

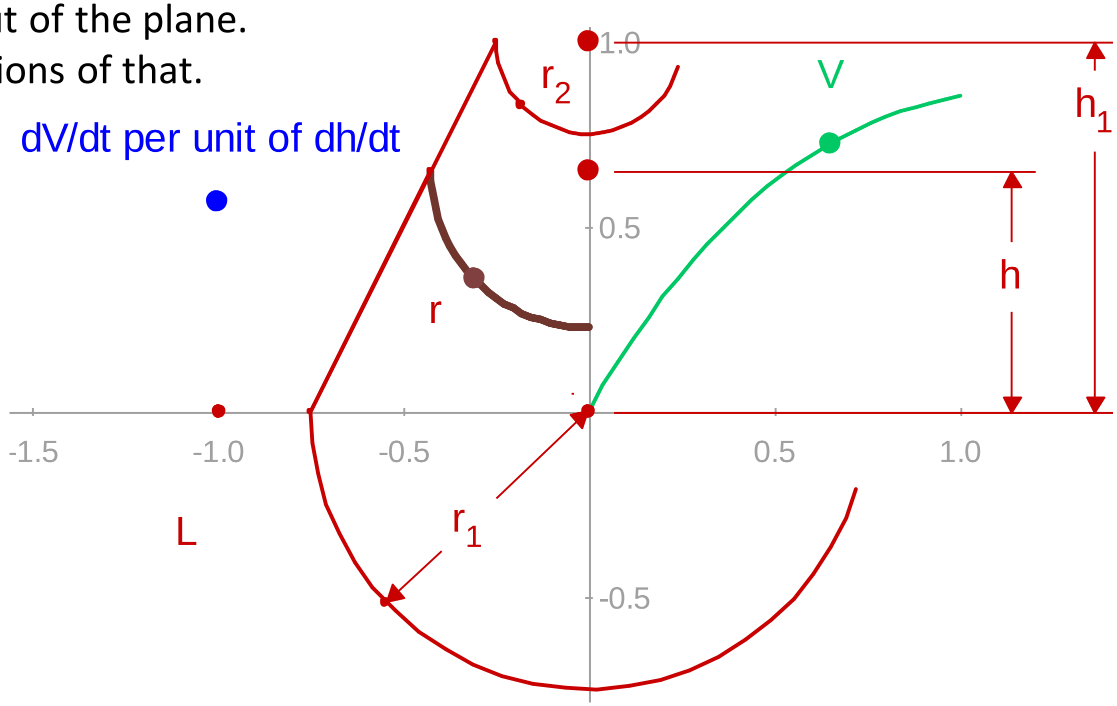
$$\frac{dV}{dt} \bigg/ \frac{dh}{dt} = \frac{dV}{dh} = \frac{\pi(r_2h - r_1h + h_1r_1)^2}{h_1^2}$$

FILLING THE CONE

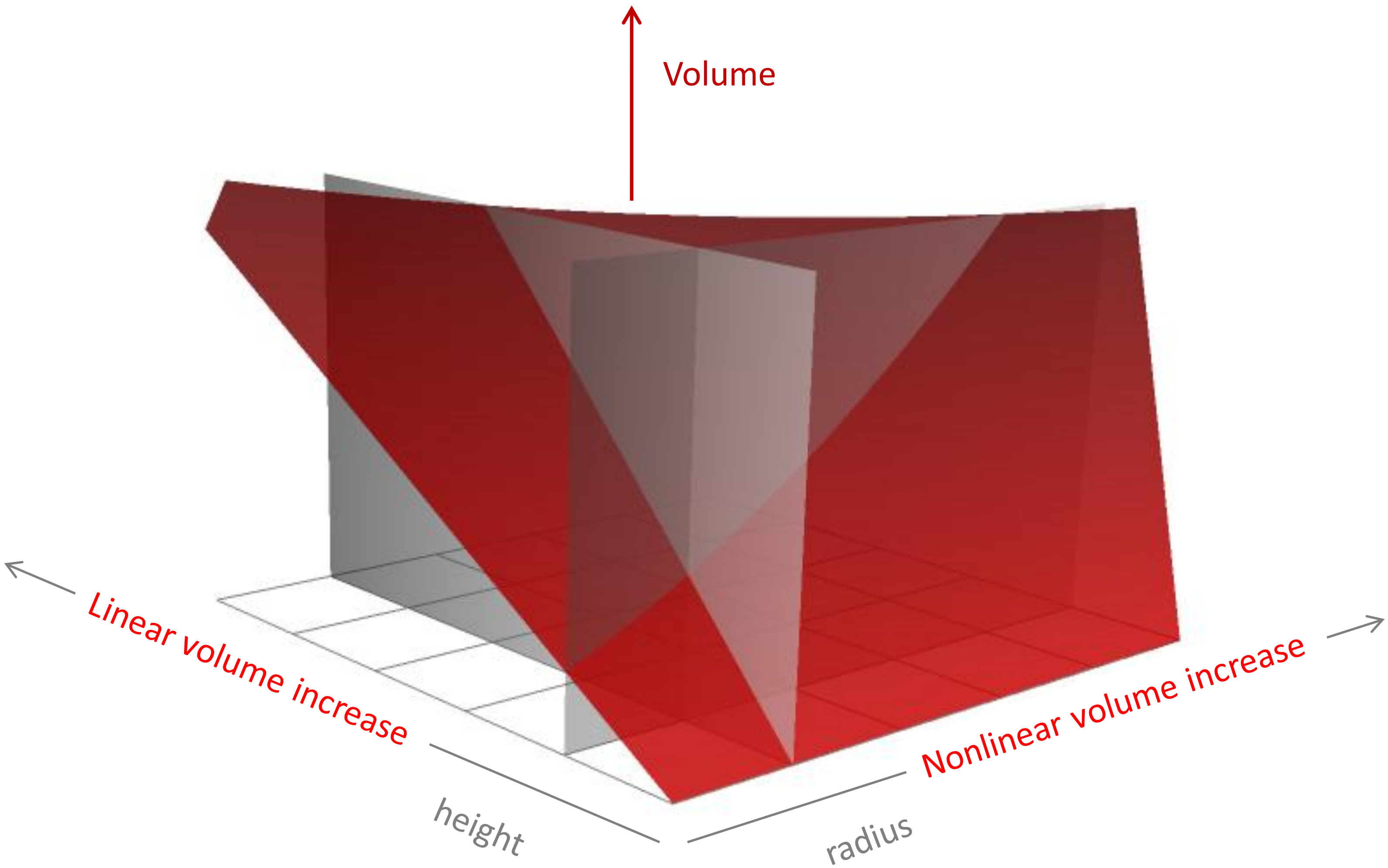
Lecture15-FillingTheCone.gx

EXERCISES:

1) Drag the height control h. Why does volume roll-off with height?
2) Animate filling the cup using h in the animation controls. How are the rates related?
3) Unlock the variables in the dialog. How does volume vary with r_1, r_2 and h_1?
4) Which increases volume more quickly, height increase or radius increase?
5) Click View→Show All.
6) This example comes out of the plane.
   Write down the limitations of that.



Filling the Cone



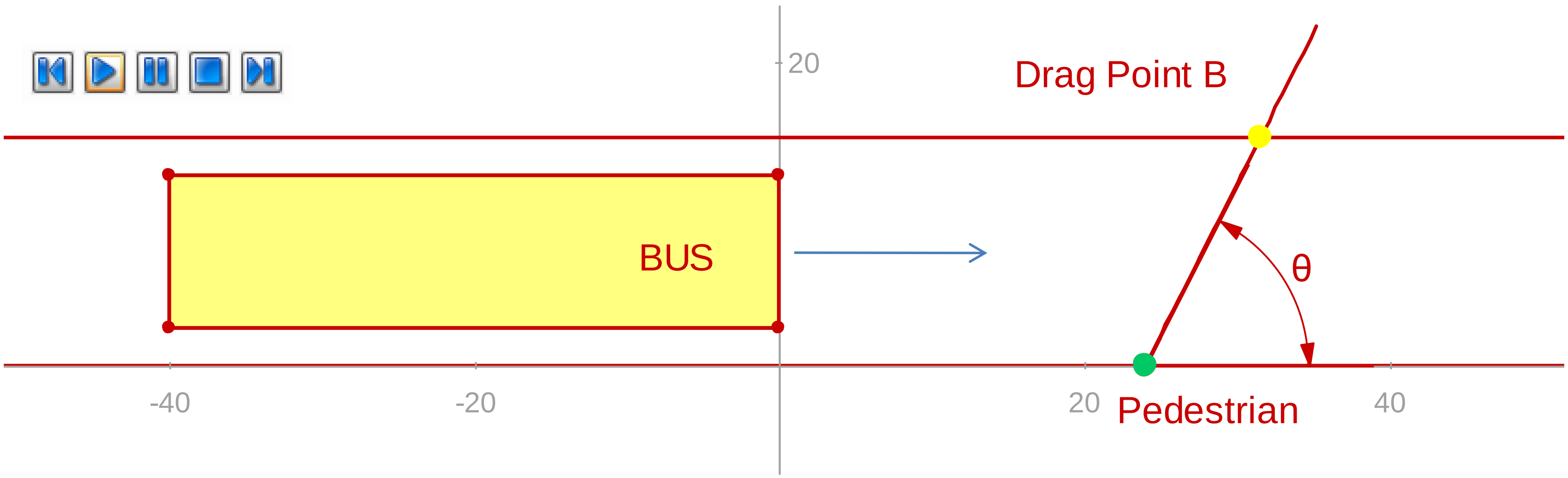
Lecture15-PedestrianVsBus.gx

Related Rates: Pedestrian vs. Bus

Consider a pedestrian late for a class, and a busdriver tired of jaywalkers. Should the pedestrian walk straight across the street or should they angle their path to avoid being "hit by the bus"?

Exercises:

1) Click the parameter t and animate the bus using the VCR controls.
2) Drag Point B and run the simulation for several values of θ.
3) Find values of θ, where the student escapes, and where they are hit.
4) How does the answer depend on the speed of the bus and pedestrian?



Related Rates: Surface Area vs. Volume

The surface area and volume of a sphere are given by:

$$A = 4\pi r^2 \quad and \quad V = \frac{4}{3}\pi r^3$$

The volume per unit of surface area is:

$$\frac{V}{A} = \frac{4}{3}\pi r^3 \Big/ 4\pi r^2 \qquad = \frac{r}{3}$$

Find the rate at which volume grows per unit of surface area.

$$dV \big/ dA = \frac{r}{3} + \frac{A}{3} \cdot \frac{dR}{dA} \quad but \quad \frac{dR}{dA} = 1/(8\pi r)$$

$$dV \big/ dA = \frac{r}{3} + \frac{4\pi r^2}{3} \cdot \frac{1}{8\pi r} = \frac{r}{3} + \frac{1}{3} \cdot \frac{4\pi r^2}{8\pi r} = \frac{r}{3} + \frac{r}{6} = \frac{r}{2}$$

Related Rates: Surface Area vs. Volume

Sphere surface area and rate of change are given by:

$$A = 4\pi r^2 \quad \text{and} \quad \frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

The volume of a sphere is given by:

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Find the rate at which volume grows per unit of surface area.

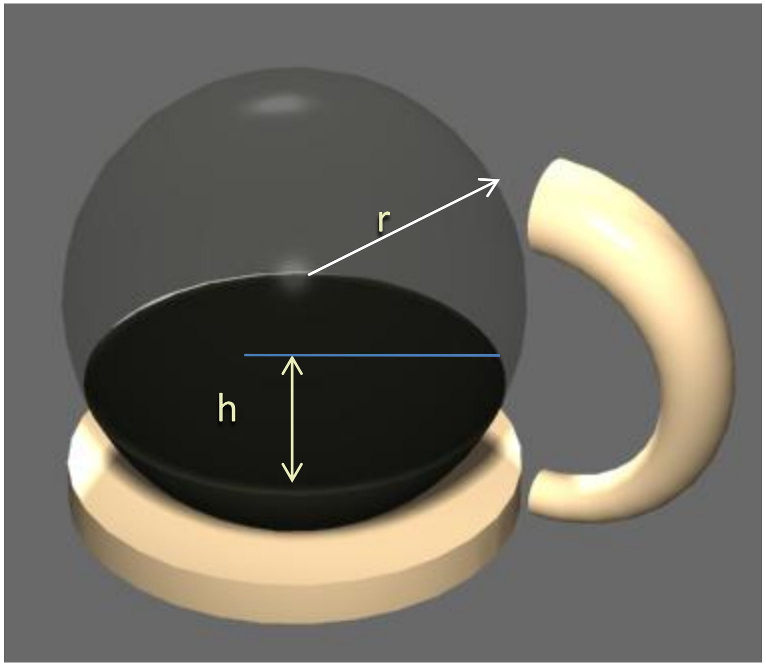
$$dV/dA = \frac{dV/dt}{dA/dt} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$$

So volume grows by a factor of r/2 faster than surface area.

Related Rates: Spheres

For a sphere fluid volume ***V*** and fluid height ***h*** are related by:

$$V_{coffee} = \frac{1}{3}\pi h^2(3r - h)$$



Differentiating with respect to h:

$$\frac{dV}{dh} = \pi \cdot h \cdot (2r - h)$$

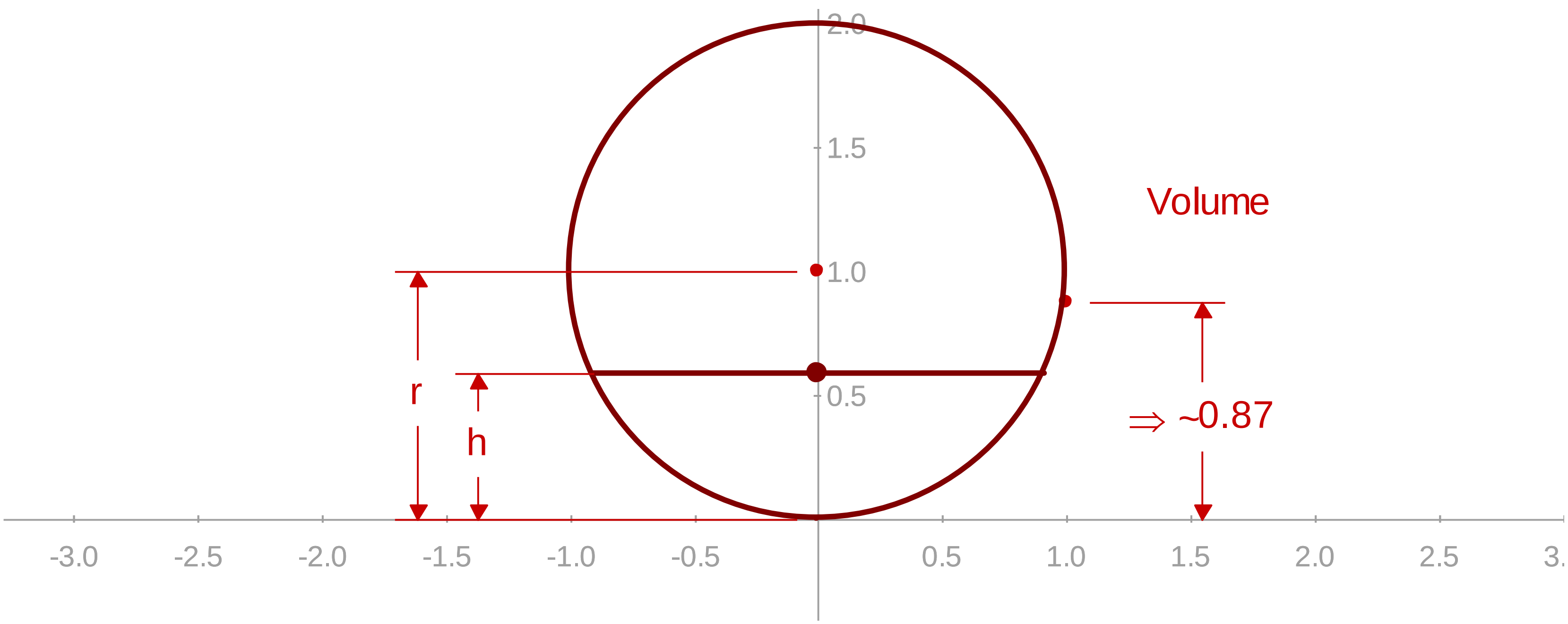
When h = 0, dV/dh = 0 and
when h = 2r, dV/dh = 0.

Filling the Sphere

Lecture15-FillingTheSphere.gx

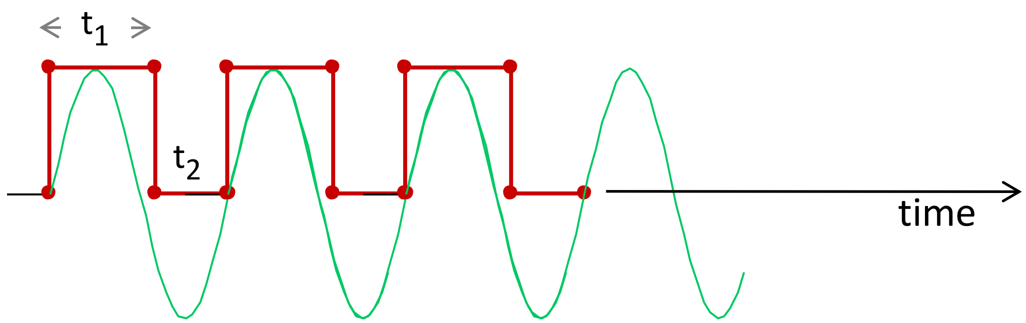
EXERCISES:

1) Simulate filling a sphere by dragging the height control h.
2) Record h when volume change is fastest.
3) Record h when volume change is slowest.
4) Modify the example to draw the dV/dh as a function of h using the x-axis for h and the y-axis for dV/dh.
5) Does this curve agree with your observations?
6) Check your work by clicking View→Show All



Related Rates: The 555 Integrated Circuit

One of the most popular electronic chips is the 555 integrated circuit. It is used to produce square waves that look like this:



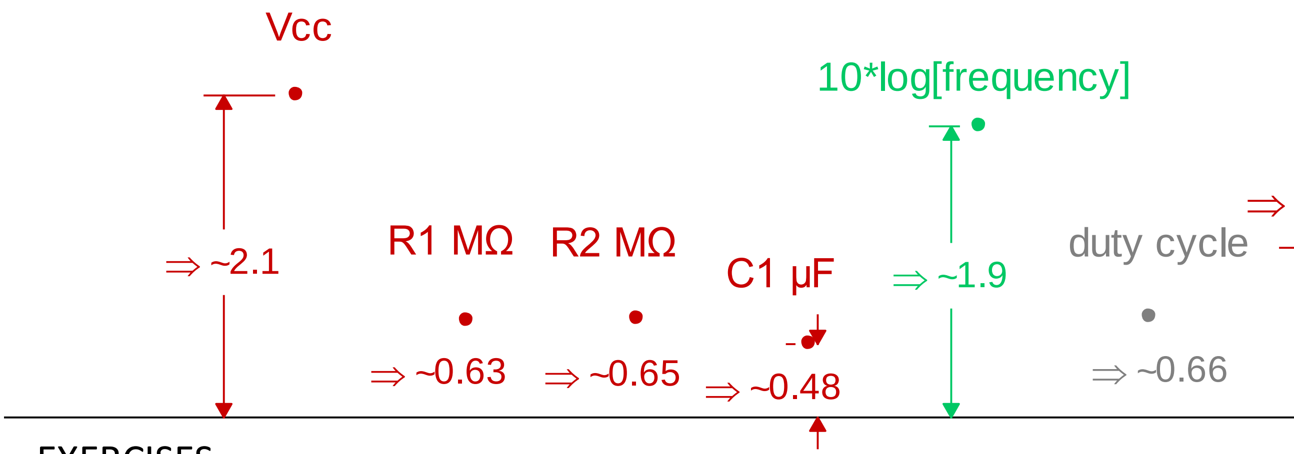
This is done by choosing the value of two resistors R_1, R_2 and one capacitor, C_1 that define the high and low pulse-widths, t_1 and t_2. The pulse widths are related by the formulas:

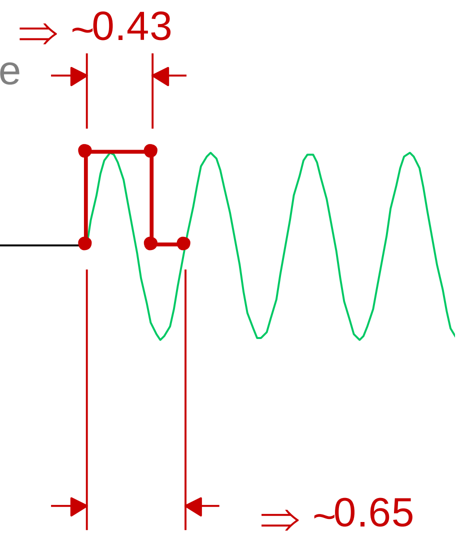
$$t_1 = \ln(2)\cdot(R_1 + R_2)\cdot C_1 \quad \text{and} \quad t_2 = \ln(2)\cdot R_2 \cdot C_1$$

We can differentiate these formulas with respect to R_1, R_2, and C_1 to determine their rate of change against component value.

THE 555 INTEGRATED CIRCUIT TIMER

Lecture15-The555IntegratedCircuit.gx

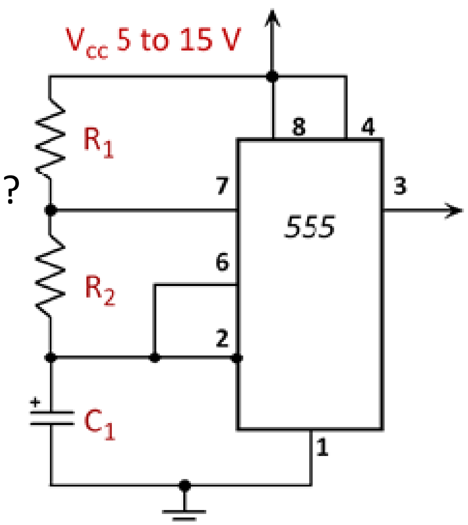




EXERCISES:

Drag red dots to change circuit values.

1) How does the square wave change?
2) What combination of values produces a 75% duty cycle?
3) Differentiate the t_1 and t_2 expressions with respect to R_1, R_2 and C_1.



Differentials Follow Same Rules As Derivatives

$$
\begin{aligned}
&d(u+v) = du + dv \\
&d(uv) \quad = udv + vdu \\
&d(u^n) \quad = nu^{n-1}du \\
&d(\sin u) = \cos u \, du \\
&etc.
\end{aligned}
$$

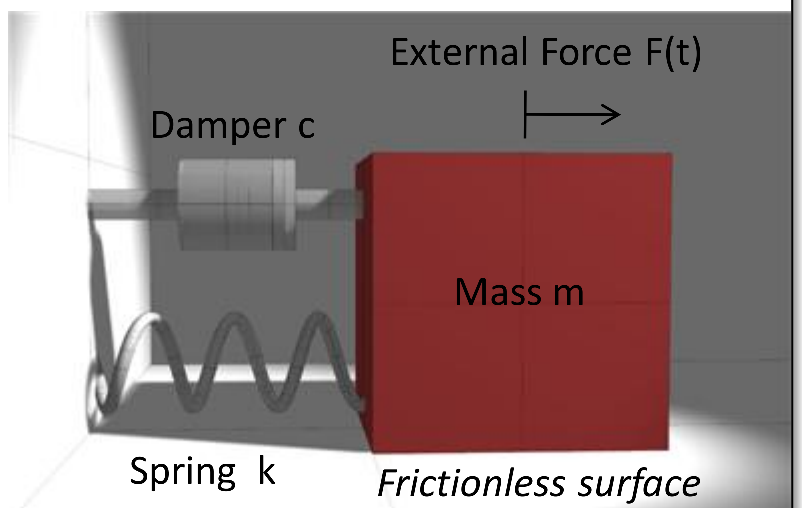
Rates Are Related By Differential Equations

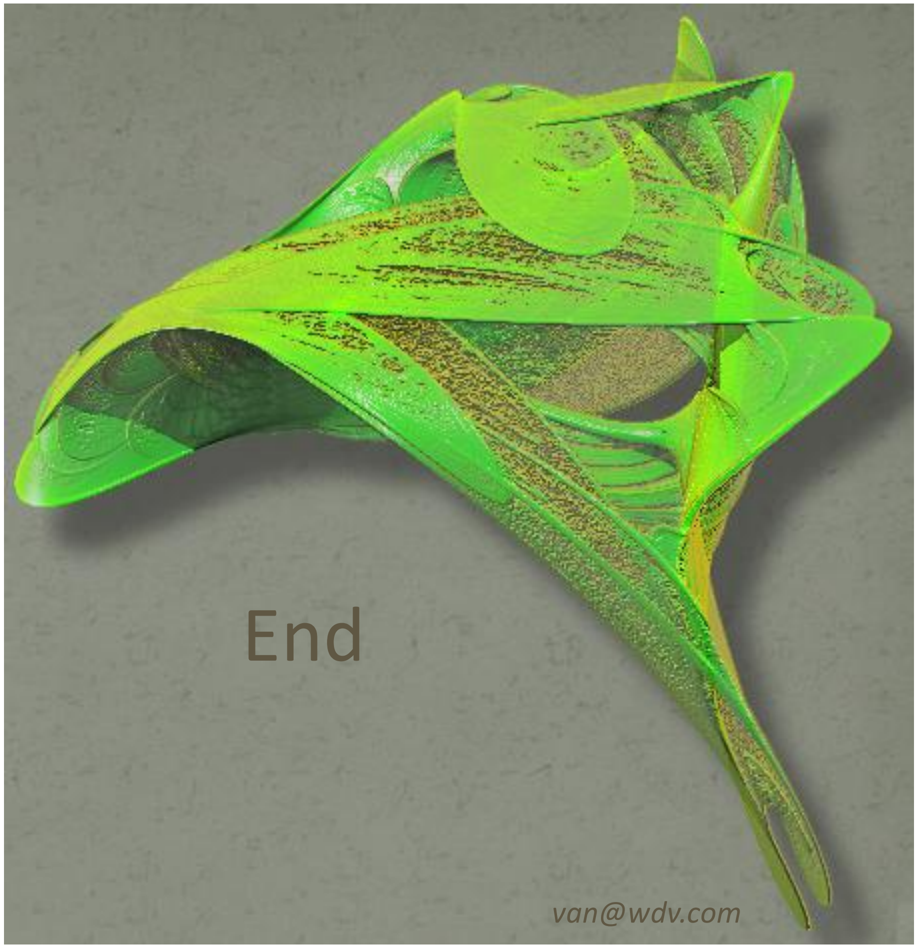
Drag Equation

$$m\ddot{x} + c\dot{x}^2 = mg$$

Second Order Linear Differential Equation

$$m\ddot{x} + c\dot{x} + kx = F(t)$$




End
van@wdv.com