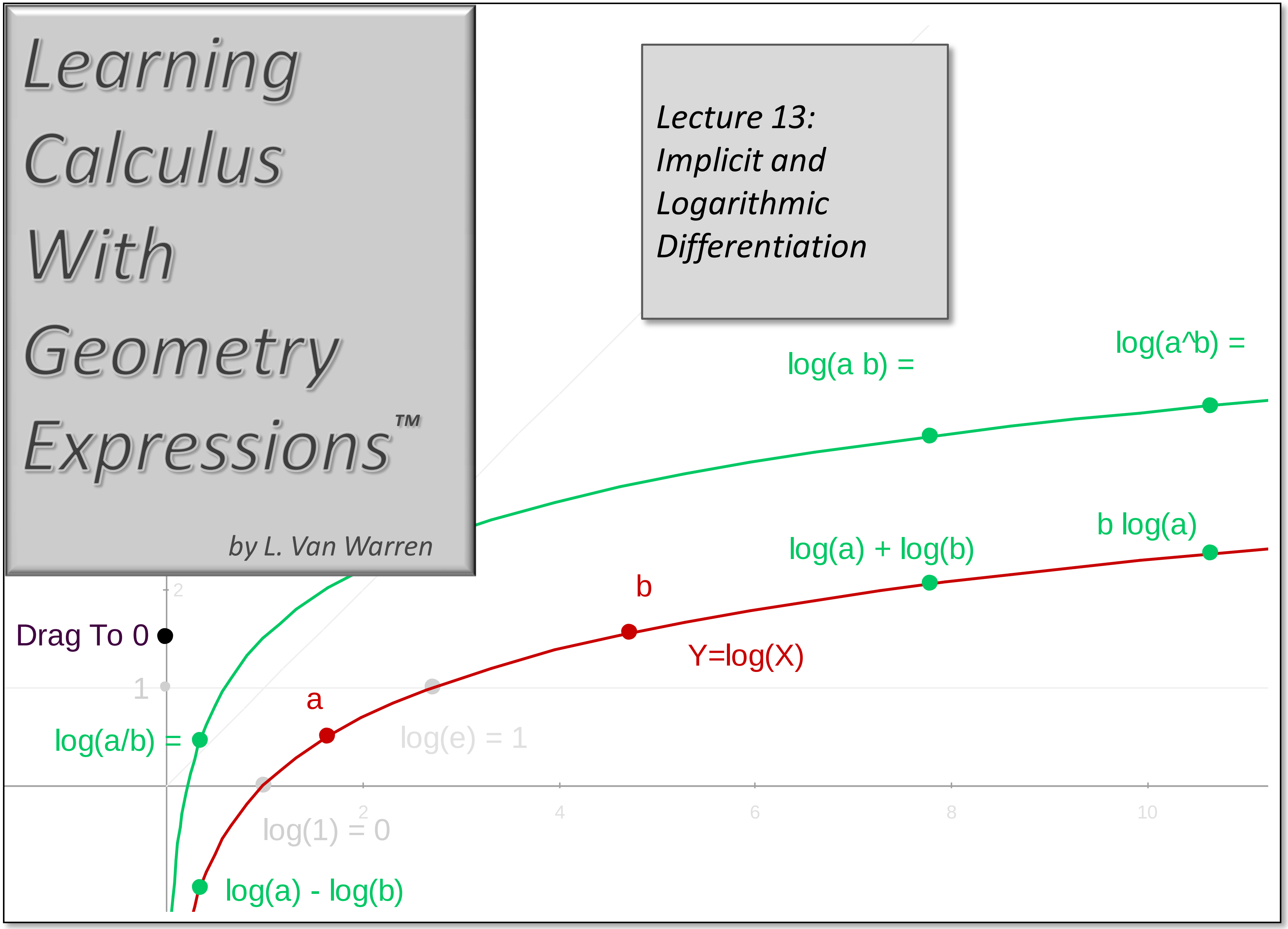
Learning Calculus With Geometry Expressions™

by L. Van Warren

Lecture 13: Implicit and Logarithmic Differentiation



Chapter 3: Derivatives

Lecture	**Topic**
10	*Definition of the Derivative*
11	*Properties of the Derivative*
12	*Derivatives of Common Functions*
13	***Implicit and Logarithmic Differentiation***

Calculus Inspiration

Brian Greene
PhD. Oxford University

- String Theory
- Quantum Gravity
- Mirror Symmetry in Calabi-Yau Manifolds

- Author of :
  - *The Elegant Universe*
  - *Icarus at the Edge of Time*
  - *The Fabric of the Cosmos*

Implicit Differentiation

Up to now we have differentiated **explicit** functions, described in Lecture 1 as separable functions where ***y*** is a function of ***x***, ***r*** is a function of θ and so forth. For this case we discovered that:

$$\frac{dy}{dx} = \frac{d}{dx}(f(x))$$

The properties of derivatives were developed and applied to f(x) in varied forms. However, ***many*** interesting curves are **implicit**, that is, we cannot separate the variables to obtain all the x's on the right and y's on the left. When this is the case we use a variation of the chain rule to find the derivative. For example consider the implicit equation for the circle:

$$x^2 + y^2 - r^2 = 0$$

Sketching our solution we implicity differentiate both sides and use the chain rule to obtain:

$$d/dx\,(x^2+y^2- r^2) = 0 \quad \rightarrow \quad d/dx\,(x^2) + d/dx\,(y^2) - d/dx\,(r^2) = 0 \rightarrow$$

$$2x\;\; dx/dx + 2y\; dy/dx = 0 \rightarrow x + y\; dy/dx = 0 \rightarrow dy/dx = -x/y$$

IMPLICIT DIFFERENTIATION

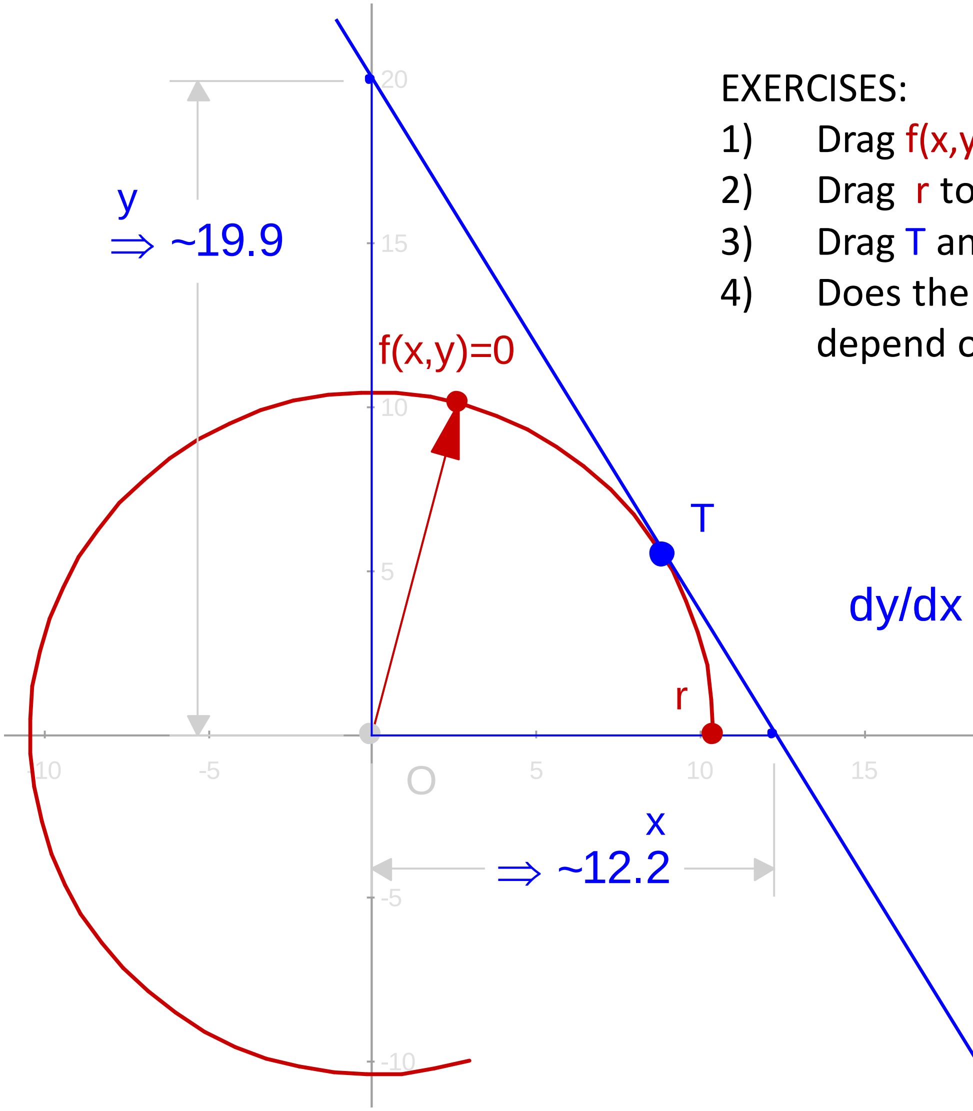
Lets repeat that symbolic calculation using ***vertical coherence***, commenting each step:

$$
\begin{aligned}
d/dx\ (x^2 + y^2 - r^2) &= 0 && \text{Take Derivative Termwise} \\
d/dx\ (x^2) + d/dx\ (y^2) &= 0 && \text{r is Constant, Deriv is 0} \\
2x\ \ dx/dx + 2y\ dy/dx &= 0 && \text{Chain Rule to Each Term} \\
x\ \ dx/dx + \ y\ dy/dx &= 0 && \text{Divide Through By 2} \\
x \qquad\quad + \ y\ dy/dx &= 0 && dx/dx = 1 \\
y\ dy/dx &= -x && \text{Sub. x from Both Sides} \\
dy/dx &= -x/y && \text{Divide Both Sides By y}
\end{aligned}
$$

This same process can be applied to any implicit equation .

Implicit Derivative of Circle:

Lecture13-CircleImplicitDeriv.gx



EXERCISES:
1) Drag f(x,y) = 0 to sweep out circle.
2) Drag r to change circle size.
3) Drag T and observe the values of -x/y.
4) Does the slope of the tangent line depend on the radius of the circle?

$$dy/dx = -x/y = -1.6310414$$

Implicit Differentiation of Bifolium

Open and run this file in wxMaxima™ to find the slope of the implicit bifolium curve.

```
(%i1) bifolium: (x^2+y^2)^2-4*a*x^2*y = 0;
```

(%o1) $\left(y^2+x^2\right)^2-4\,a\,x^2\,y=0$

```
(%i2) depends(y,x)$ /* The trick. */
```

```
(%i3) slope: diff(bifolium,x);
```

(%o3) $2\left(y^2+x^2\right)\left(2\,y\left(\frac{d}{d\,x}\,y\right)+2\,x\right)-4\,a\,x^2\left(\frac{d}{d\,x}\,y\right)-8\,a\,x\,y=0$

```
(%i4) solve(slope, 'diff(y,x));
```

(%o4) $[\frac{d}{d\,x}\,y=-\frac{x\,y^2-2\,a\,x\,y+x^3}{y^3+x^2\,y-a\,x^2}]$

The resulting slope is an implicit equation in ***x*** and ***y***!

Lecture13-ImplicitDerivBifoliumPolar.wxm

Implicit Derivative in Polar Coordinates

Perhaps the derivative in polar coordinates will be explicit, running the next file gives us:

```
(%i1) dydx: -(x*y^2-2*a*x*y+x^3)/(y^3+x^2*y-a*x^2);
```

(%o1) $$\frac{-x\,y^2+2\,a\,x\,y-x^3}{y^3+x^2\,y-a\,x^2}$$

```
(%i2) ratsubst(r*cos(t), x, ratsubst(r*sin(t), y, dydx));
```

(%o2) $$-\frac{r\cos(t)\sin(t)^2-2\,a\cos(t)\sin(t)+r\cos(t)^3}{r\sin(t)^3+r\cos(t)^2\sin(t)-a\cos(t)^2}$$

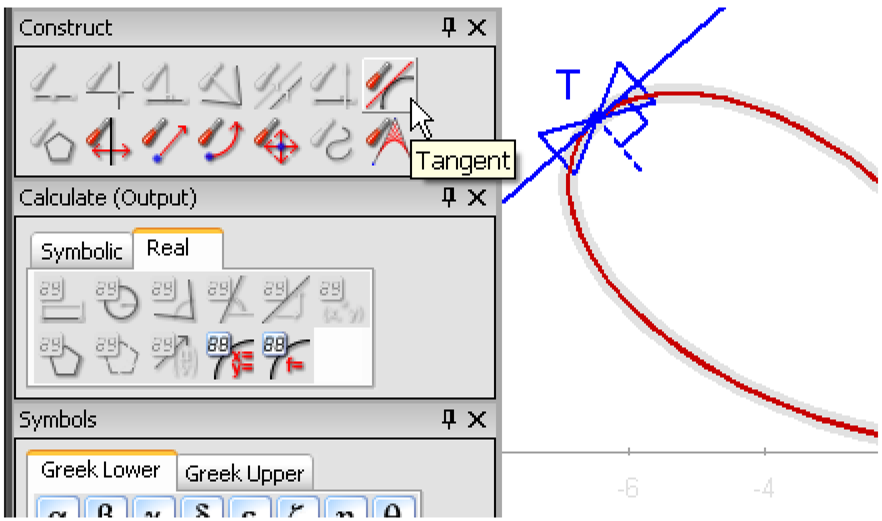
```
(%i3) trigsimp(%);
```

(%o3) $$\frac{2\,a\cos(t)\sin(t)-r\cos(t)}{r\sin(t)-a\cos(t)^2}$$

Again dy/dx is a function of both ***r*** and θ, which is still implicit.

Bifolium Tangent Line… Automatically

Since we can define the bifolium as an explicit function of θ we can draw it in Geometry Expressions™. We can then select the curve, choose the tangent icon and the tangent line is computed and displayed automatically. Try it yourself.



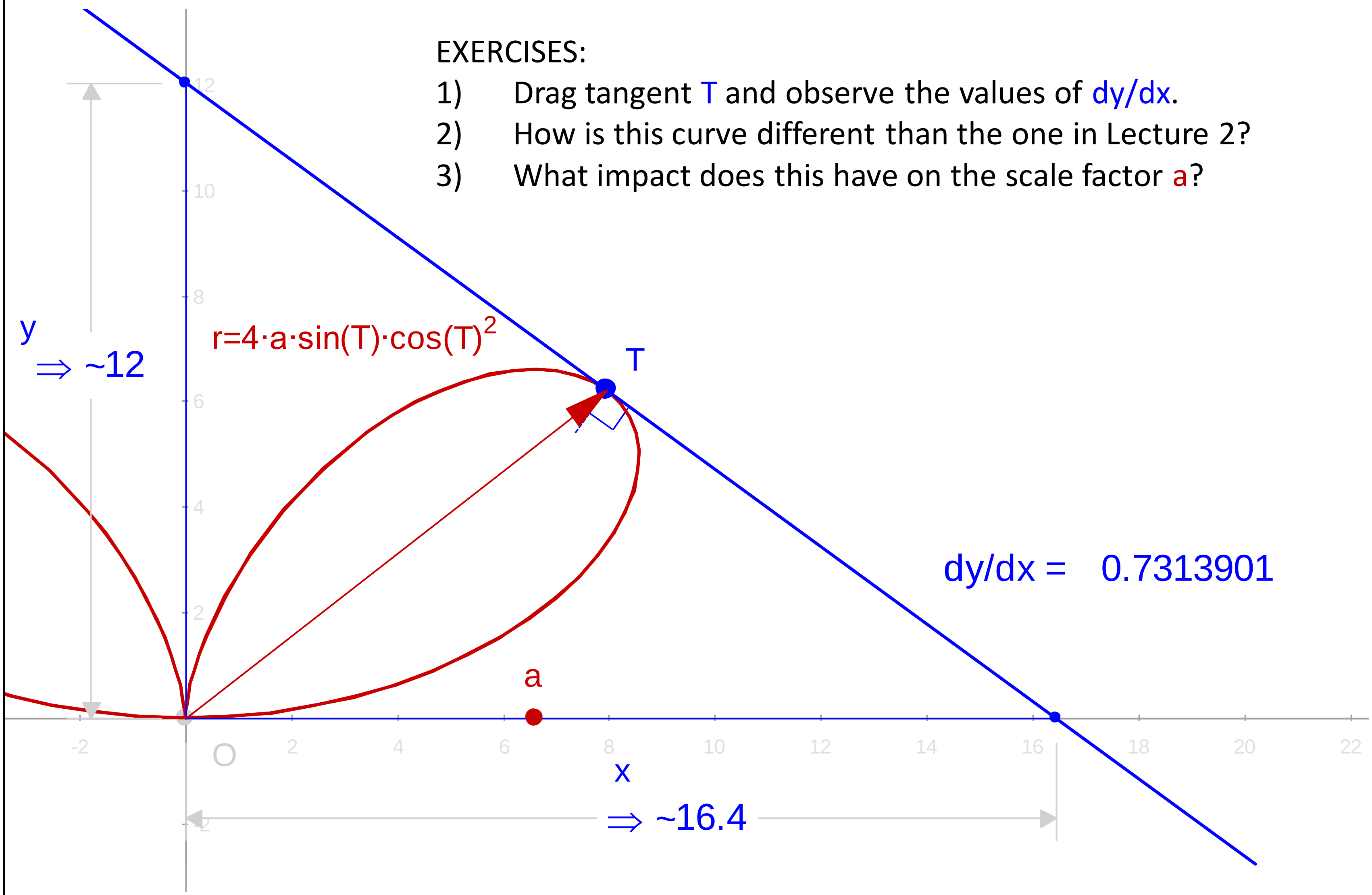
For the general case, the interconversion of the three represention, explicit, implicit and parmetric is a research topic in advance mathematics.

Implicit Derivative of Bifolium:

Lecture13-BifoliumImplicitDeriv.gx

EXERCISES:

1) Drag tangent T and observe the values of dy/dx.
2) How is this curve different than the one in Lecture 2?
3) What impact does this have on the scale factor a?



Implicit Derivative of Atlas Curves:

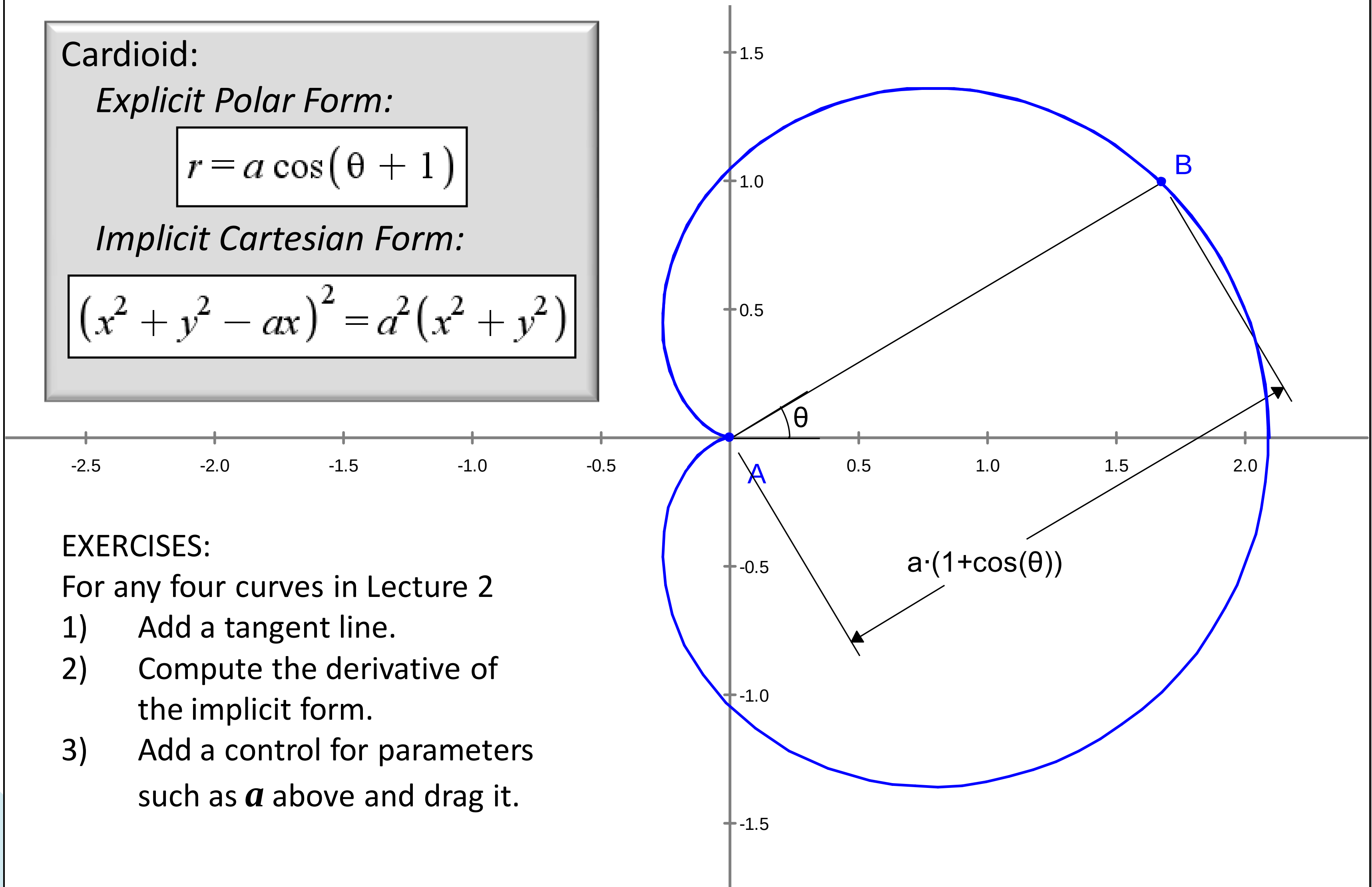
Cardioid:

Explicit Polar Form:

$$r = a\cos(\theta + 1)$$

Implicit Cartesian Form:

$$(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2)$$



EXERCISES:

For any four curves in Lecture 2

1) Add a tangent line.
2) Compute the derivative of the implicit form.
3) Add a control for parameters such as $\boldsymbol{a}$ above and drag it.

LOGARITHMIC DIFFERENTIATION

Just as the chain rule provides a tool for differentiating implicit functions, logarithmic differentiation provides a tool for differentiating exponential functions which do not fit into any of the categories we have studed. One example of this is finding the derivative of:

$$y = x^x$$

The power rule will not work for this because in the equation:

$$y = x^n$$

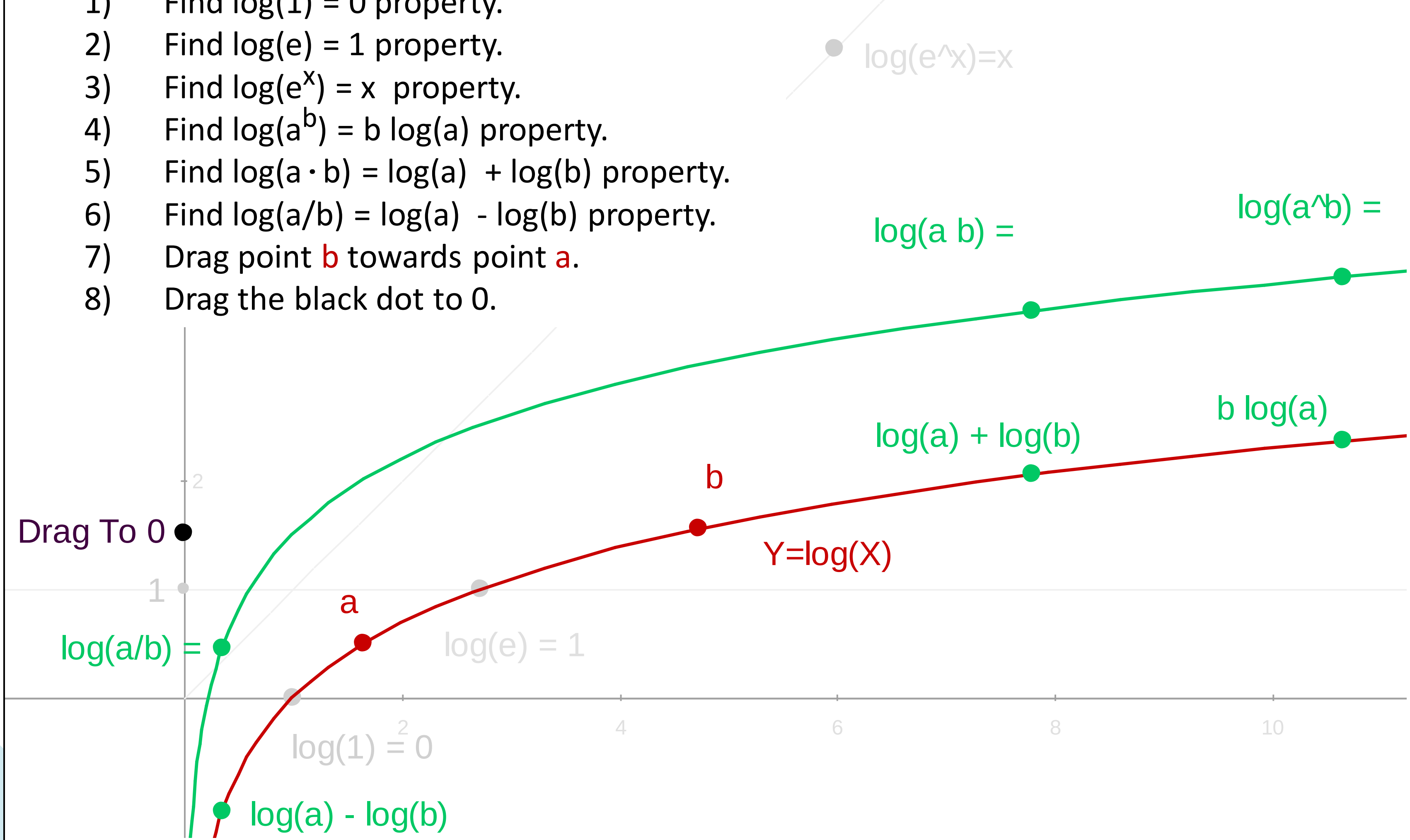
The exponent ***n*** is considered constant. The solution is logarithmic differentation, but for that to work, we must first review the properties of logarithms and we will do that with Geometry Expressions™. Note that natural log is written as log(x). Other sources may use ln(x) for natural log.

Lecture13-LogProperties.gx

Properties of Logarithms

EXERCISES:

1) Find log(1) = 0 property.
2) Find log(e) = 1 property.
3) Find $\log(e^x) = x$ property.
4) Find $\log(a^b) = b \log(a)$ property.
5) Find $\log(a \cdot b) = \log(a) + \log(b)$ property.
6) Find log(a/b) = log(a) - log(b) property.
7) Drag point b towards point a.
8) Drag the black dot to 0.



Logarithmic Differentiation Of An Exploding Function

Now that we have the properties of logarithms in hand, let's find the derivative of:

$$y = x^x$$

The trick is to take the log of BOTH SIDES, and use the property of logs:

$$log(y) = log(x^x) = x\,log(x)$$

Now we differentiate both sides using the product rule to obtain:

$$\frac{1}{y} \cdot y' = 1 + log(x)$$

Multiplying both sides by *y* yields:

$$y' = y(1 + log(x))$$

Back substituting the original function of *y* provides the explicit result:

$$y' = x^x(1 + log(x))$$

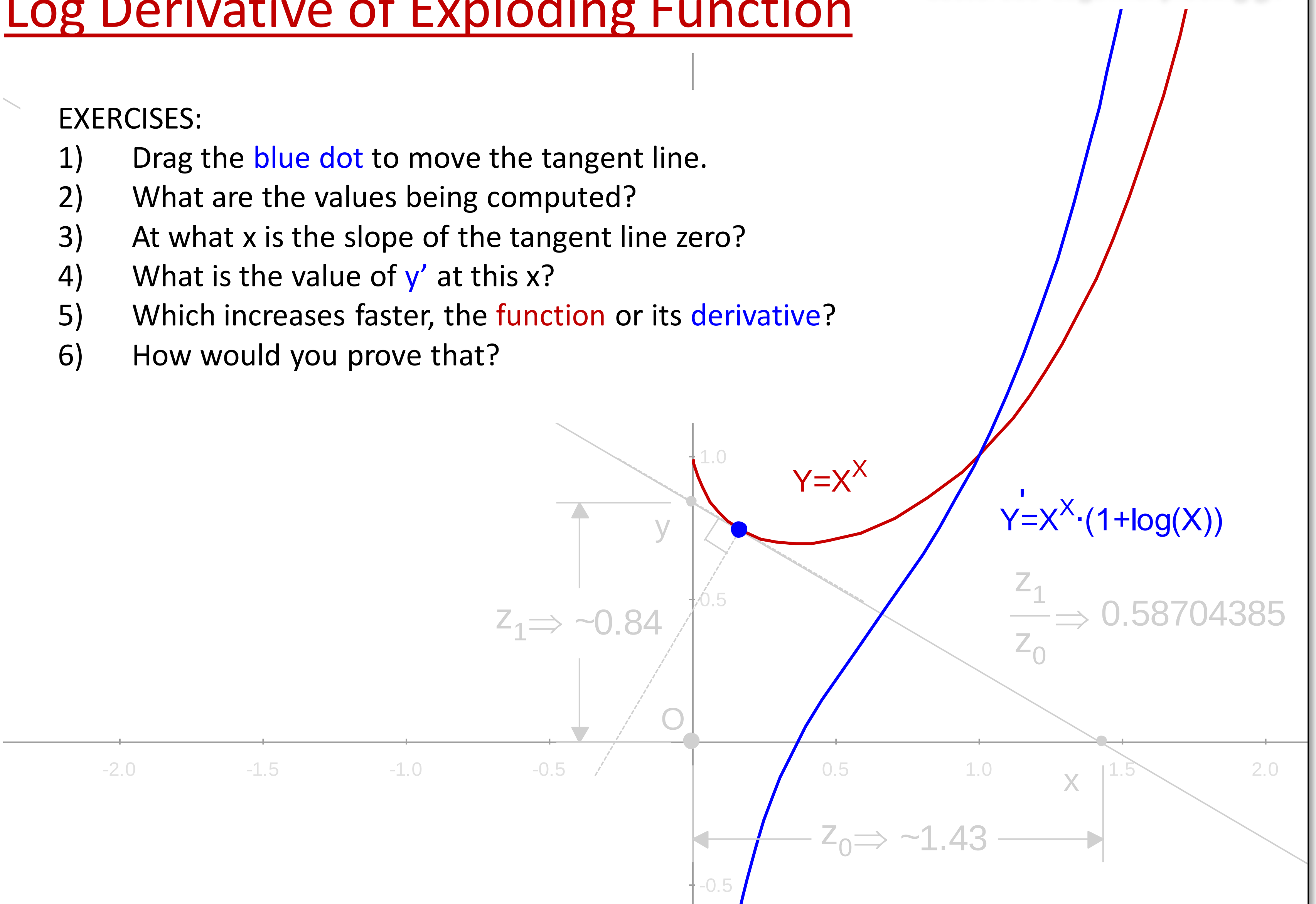
Which we can now examine in Geometry Expressions™.

Log Derivative of Exploding Function

Lecture13-LogDiffExploding.gx

EXERCISES:

1) Drag the blue dot to move the tangent line.
2) What are the values being computed?
3) At what x is the slope of the tangent line zero?
4) What is the value of y' at this x?
5) Which increases faster, the function or its derivative?
6) How would you prove that?



Logarithmic Differentiation Of A Slow Function

Functions like $\frac{1}{x}, \sqrt{x}$ and log(x) change slowly relative to linear functions for large values of x. We need logarithms to differentiate increasing functions like:

$$y = \log(x)^{\log(x)}$$

Using wxMaxima™ to manage expression complexity we run the problem and obtain:

```
(%i1) slowfun: log(x)^log(x);
```

(%o1) $\log(x)^{\log(x)}$

```
(%i2) diff(slowfun,x);
```

(%o2) $\log(x)^{\log(x)}\left(\frac{\log(\log(x))}{x}+\frac{1}{x}\right)$

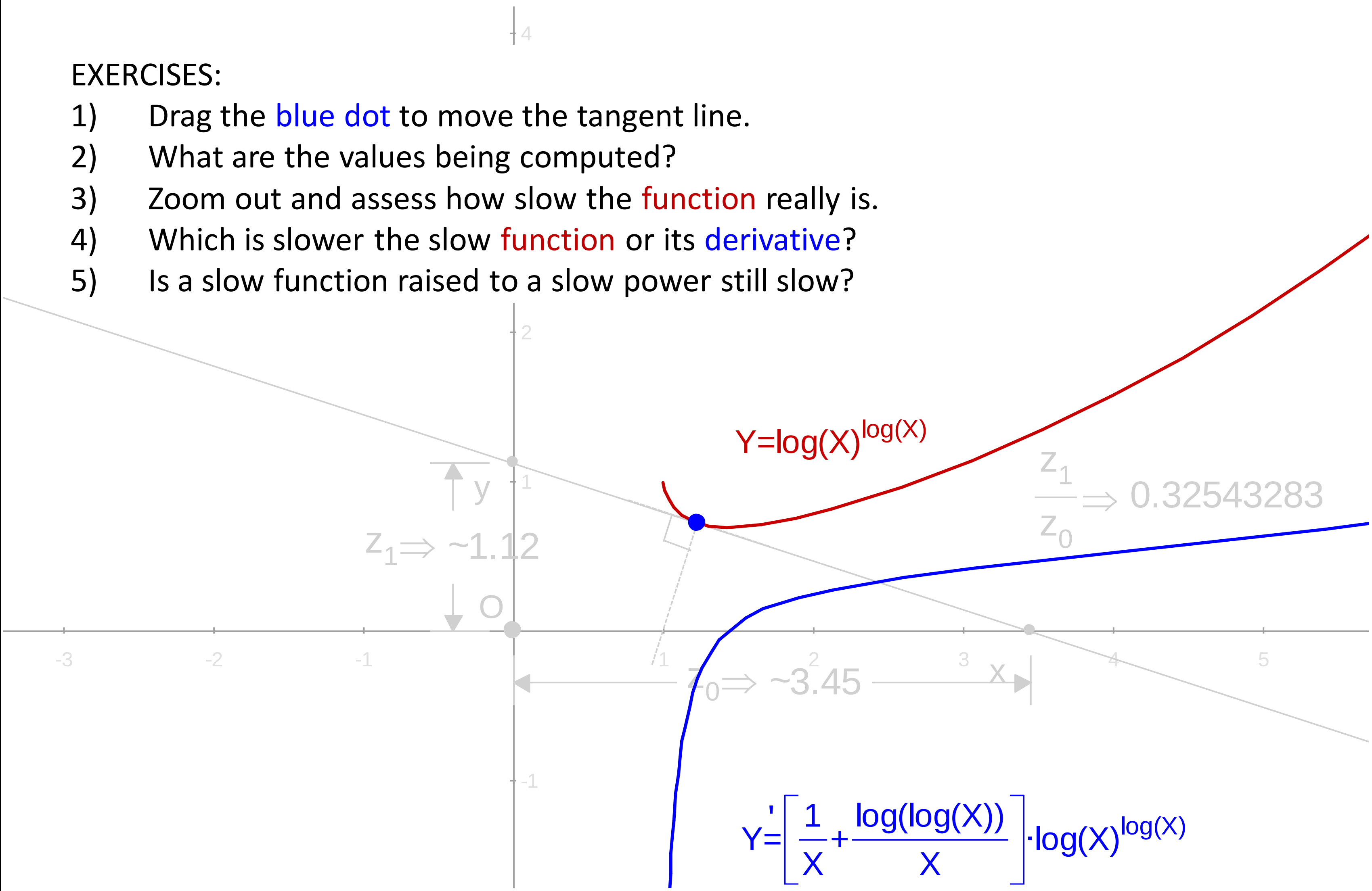
Which we can now examine in Geometry Expressions™.

Log Derivative of Slow Function

Lecture13-LogDiffSlow.gx

EXERCISES:

1) Drag the blue dot to move the tangent line.
2) What are the values being computed?
3) Zoom out and assess how slow the function really is.
4) Which is slower the slow function or its derivative?
5) Is a slow function raised to a slow power still slow?



Logarithmic Differentiation Of A Periodic Function

We can logarithmically differentiate the periodic function:

$$y = \sin(x)^{\frac{1}{x}}$$

Using wxMaxima™ to manage expression complexity we run the problem and obtain:

```
(%i1) periodic: sin(x)^(1/x);
```

(%o1) $\sin(x)^{1/x}$

```
(%i2) diff(periodic,x);
```

(%o2) $\sin(x)^{1/x}\left(\frac{\cos(x)}{x\sin(x)} - \frac{\log(\sin(x))}{x^2}\right)$

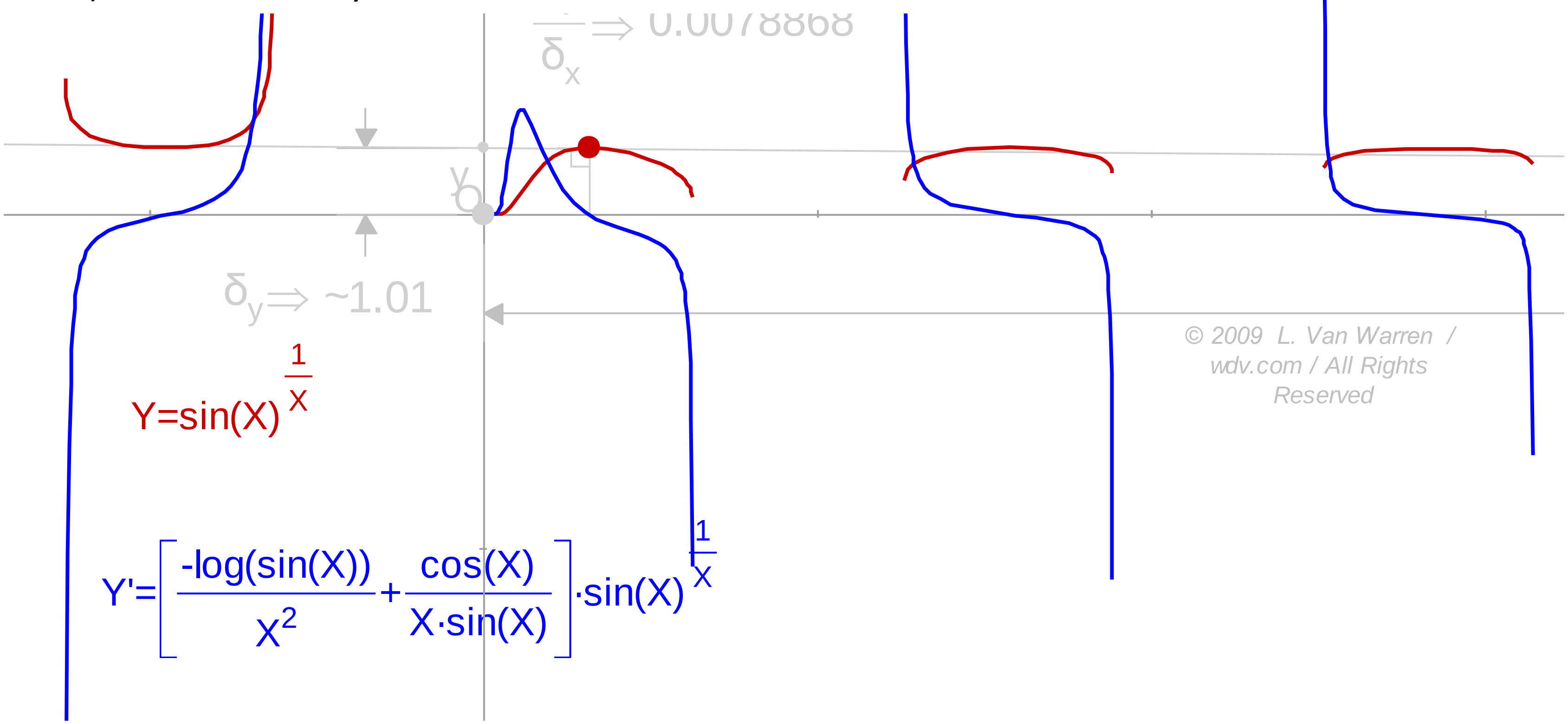
We now examine the periodic function and its derivative in Geometry Expressions™:

Log Derivative of Periodic Function

Lecture13-LogDiffPeriodic.gx

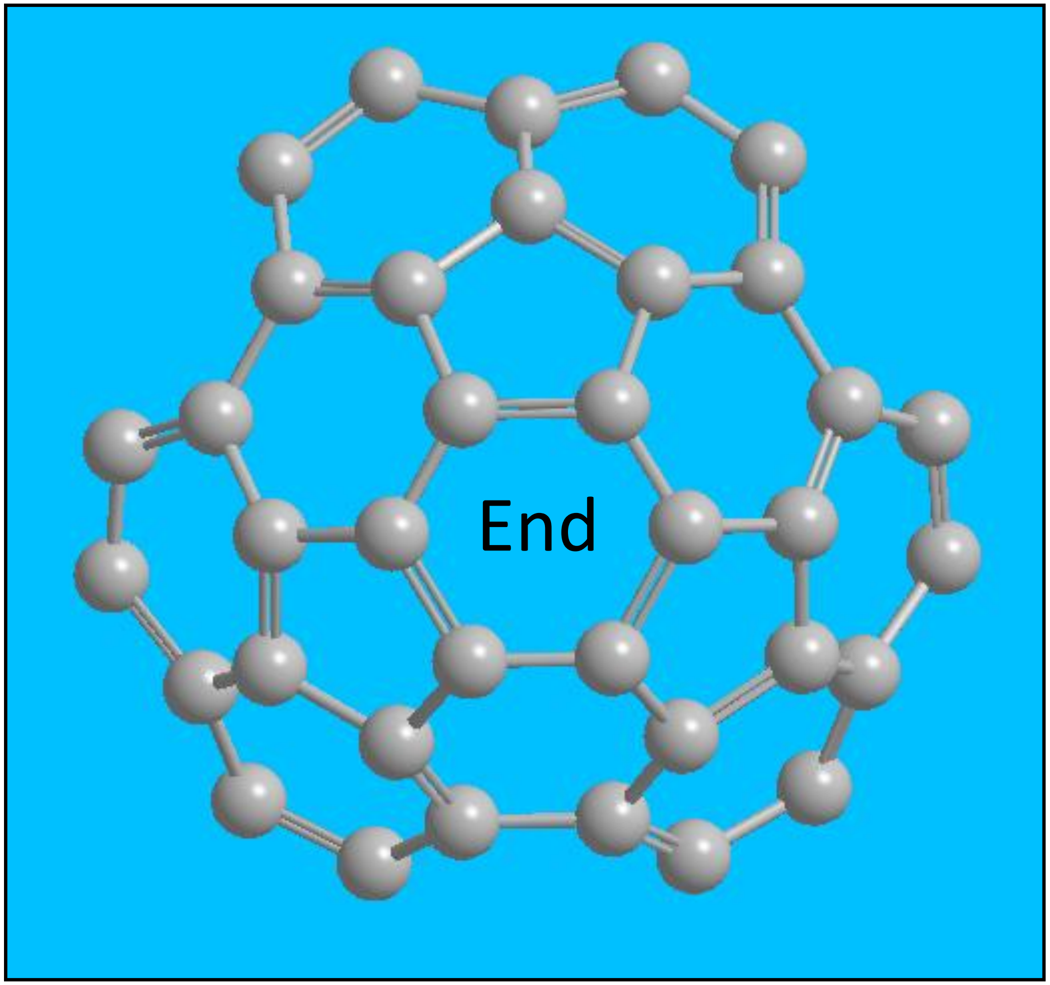
EXERCISES:

1) Drag the blue dot across different sections of the function.
2) What are the values of the function *between* sections?
3) How would you evaluate them?



$$Y=\sin(X)^{\frac{1}{X}}$$

$$Y'=\left[\frac{-\log(\sin(X))}{X^2}+\frac{\cos(X)}{X\cdot\sin(X)}\right]\cdot\sin(X)^{\frac{1}{X}}$$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