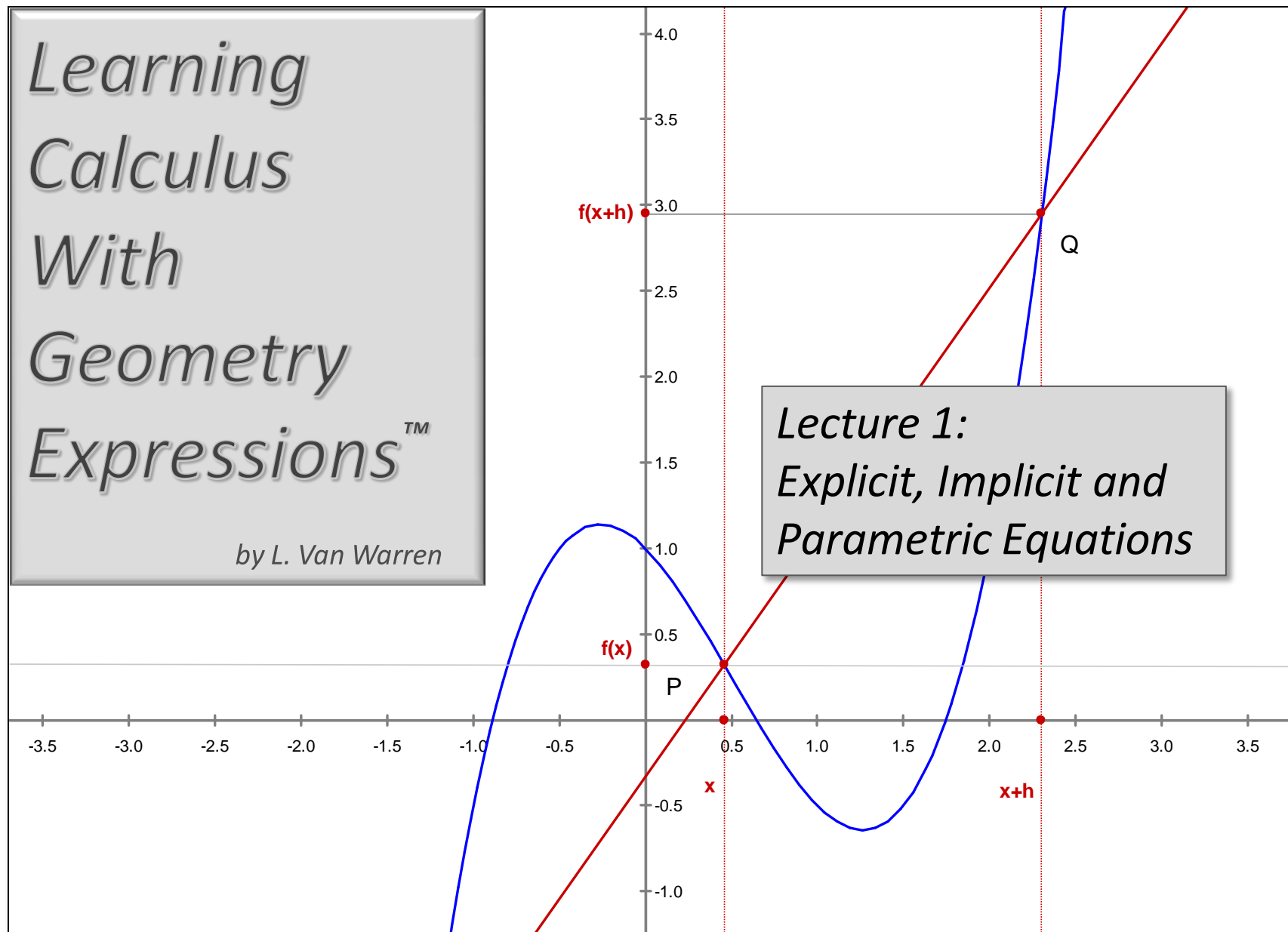


Learning Calculus With Geometry ExpressionsTM

by L. Van Warren



Chapter 1: Functions and Equations

LECTURE	TOPIC
<i>0</i>	<i>GEOMETRY EXPRESSIONS™ WARM-UP</i>
1	EXPLICIT, IMPLICIT AND PARAMETRIC EQUATIONS
<i>2</i>	<i>A SHORT ATLAS OF CURVES</i>
<i>3</i>	<i>SYSTEMS OF EQUATIONS</i>
<i>4</i>	<i>INVERTIBILITY, UNIQUENESS AND CLOSURE</i>

Calculus Inspiration

Louis Eric Wasserman

used a novel approach for attacking the [Clay Math Prize](#) of P vs. NP.

He examined problem complexity. Louis calculated the least number of gates needed to compute explicit functions using only AND and OR, the basic atoms of computation. He produced a characterization of P, a class of problems that can be solved in by computer in polynomial time.

He also likes ultimate Frisbee™.



EXPLICIT FUNCTIONS

Mathematicians like Louis use the term “explicit function” to express the idea that we have one dependent variable on the left-hand side of an equation, and all the independent variables and constants on the right-hand side of the equation. For example, the equation of a line is:

$$y = mx + b;$$
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1};$$

Where ***m*** is the slope and ***b*** is the *y*-intercept.

Explicit functions GENERATE *y* values from *x* values.

LINEAR EQUATIONS

In the linear equation above, ***m*** and ***b*** are considered constants. If we wanted we could declare ***m*** and ***b*** to be variables and *x* to be a constant. Switching the way we look at things is often useful, but for now we’ll start simply. It is important to declare definitions and assumptions from the outset to avoid mistakes.

FUNCTIONAL NOTATION

Every book that talks about linear functions has to say the following:

The values assigned to the independent variable x are the **domain** of the function.

The values of the dependent variable y are the **range** of the function.

Functions can have names other than y .

If we want to give the function a name, like f , we write:

$$y = f(x);$$

allowing the explicit form of a linear equation to be written as:

$$f(x) = mx + b;$$

When $f(x)$ has this definition, the graph is a line.

CONVENIENT FORM

The equation for a line in **convenient form** is:

$$y = a + bx;$$

Notice we have switched the order of terms and given them different names.

Now a is the y -intercept and b is the slope.

Writing an explicit equation in this way has an advantage. We could write:

$$y = a,$$

Which would mean that y is a constant, unchanging for any value of x .

We can escalate complexity in a convenient way by first writing:

$$y = a + bx;$$

And then writing the polynomial:

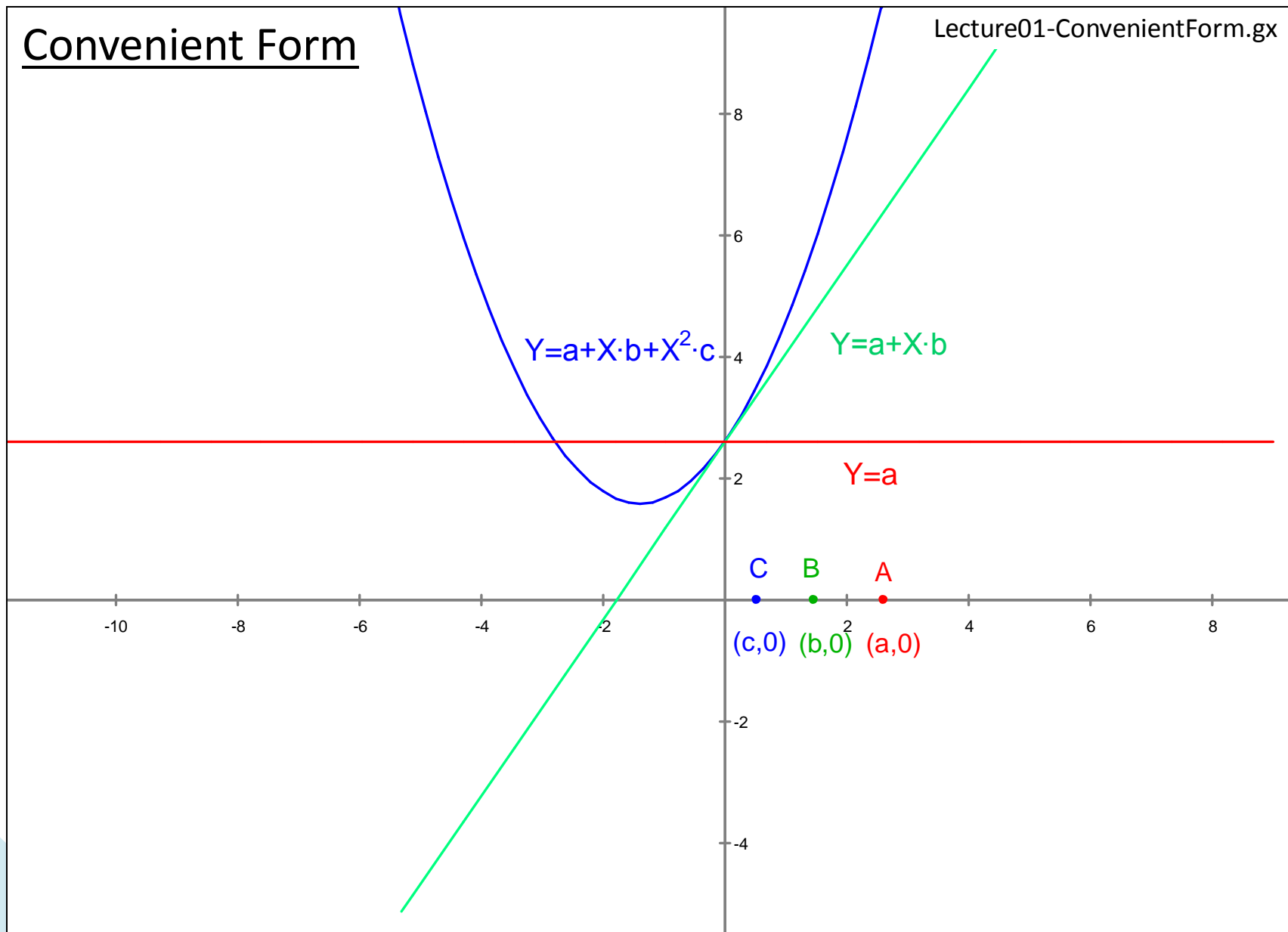
$$y = a + bx + cx^2;$$

and so on. The constants a , b , and c are called **coefficients** of the polynomial.

Be prepared to work in both traditional forms and **convenient forms** interchangeably.

Convenient Form

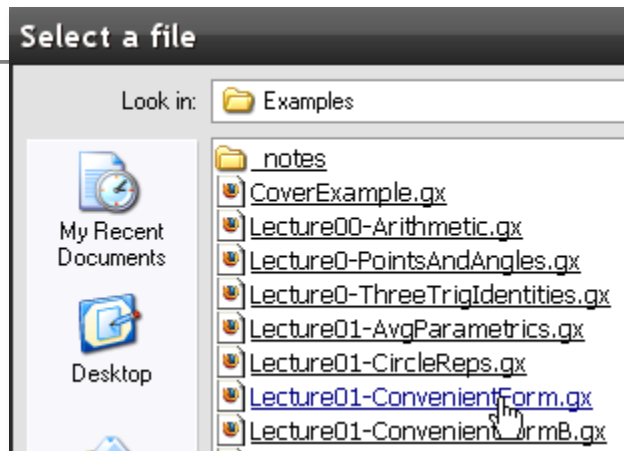
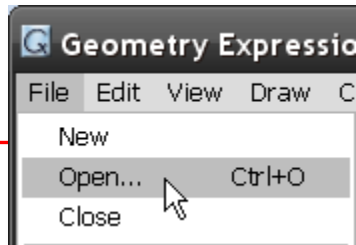
Lecture01-ConvenientForm.gx



Convenient Form

EXERCISE

- 1) Open the example.



File → Open → Lecture01-ConvenientForm.gx

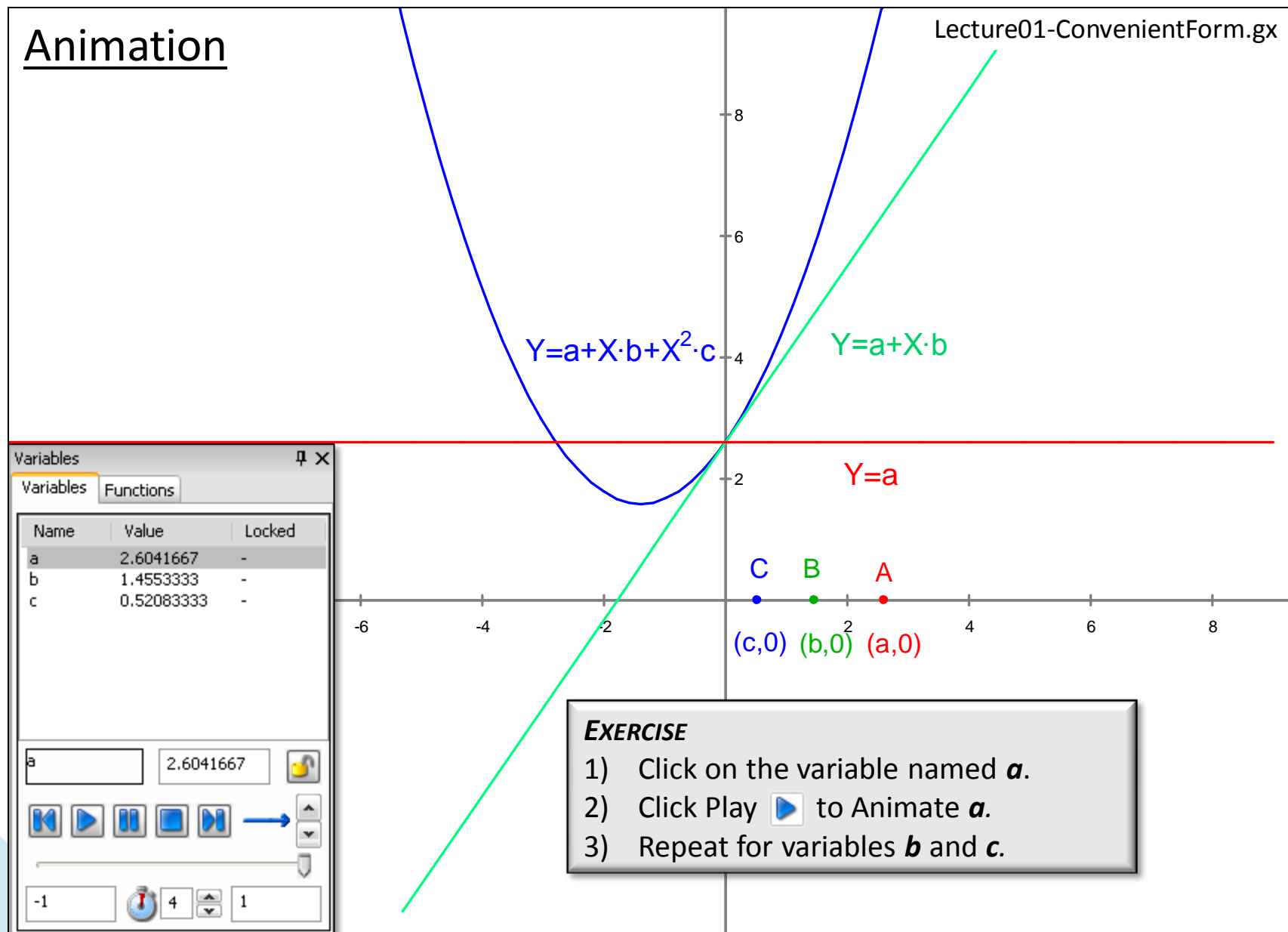
$$Y = a + X \cdot b + X^2 \cdot c$$

$$Y = a + X \cdot b$$

$$Y = a$$

C B A
 $(c,0)$ $(b,0)$ $(a,0)$

Animation



Extending An Example

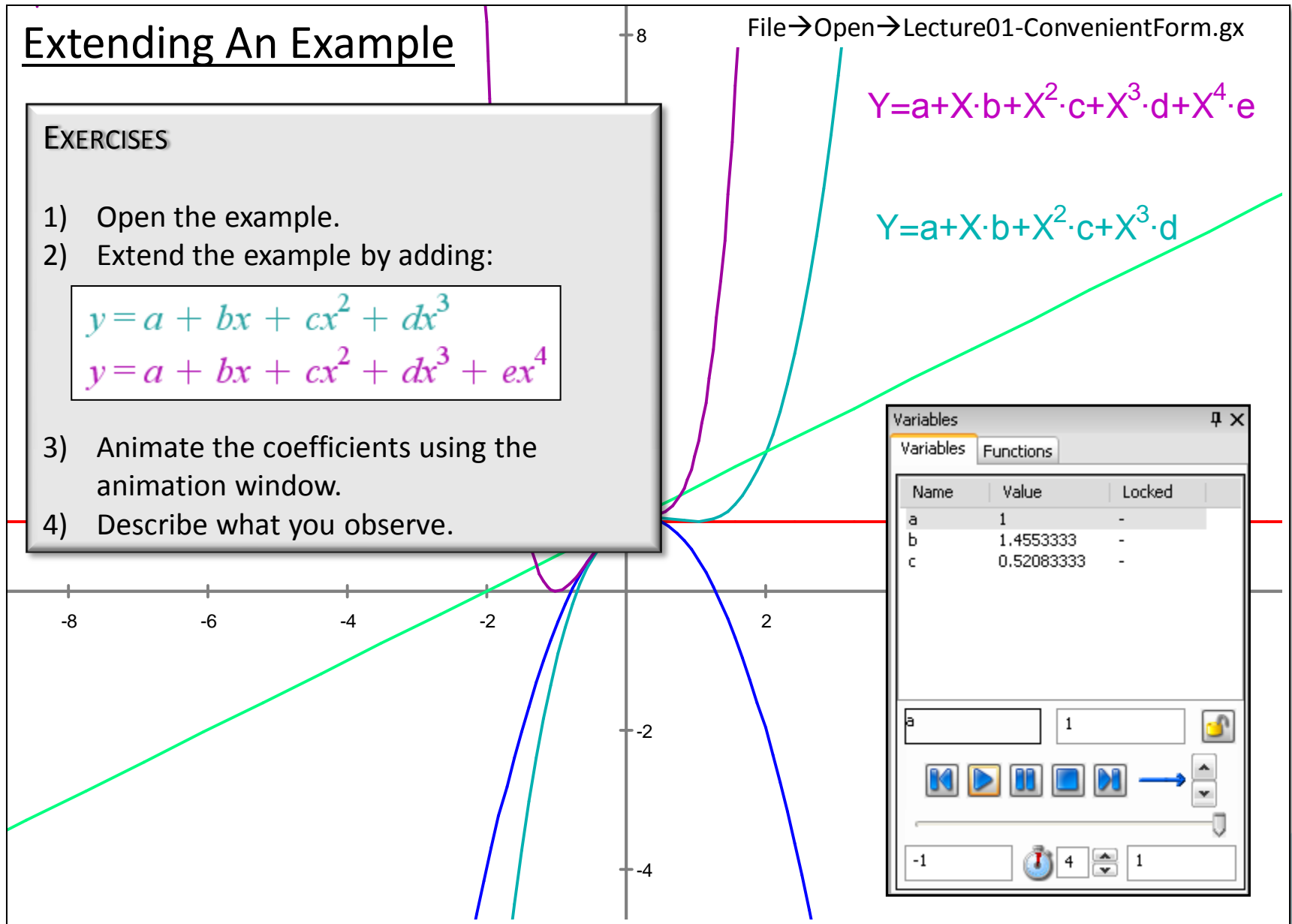
EXERCISES

- 1) Open the example.
- 2) Extend the example by adding:

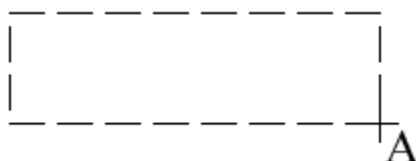
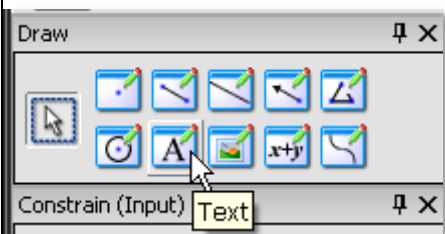
$$y = a + bx + cx^2 + dx^3$$

$$y = a + bx + cx^2 + dx^3 + ex^4$$

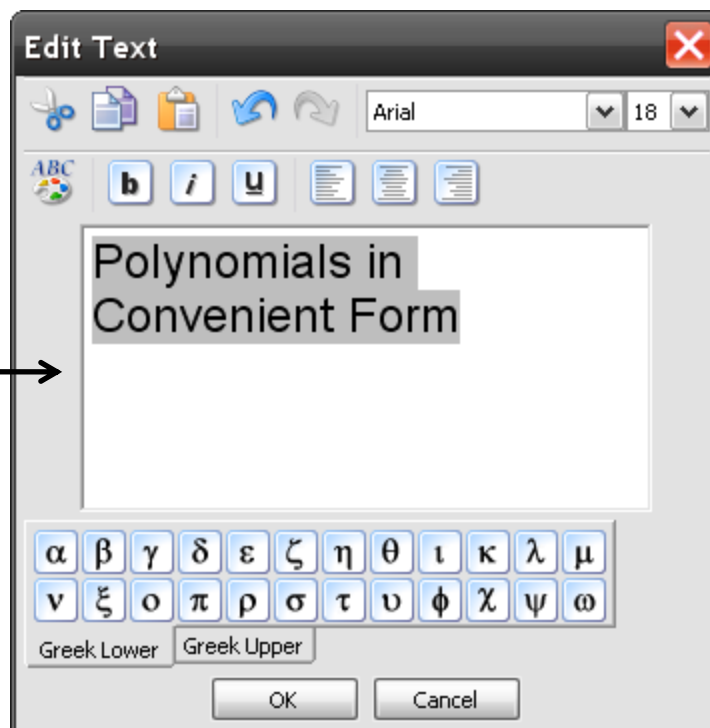
- 3) Animate the coefficients using the animation window.
- 4) Describe what you observe.

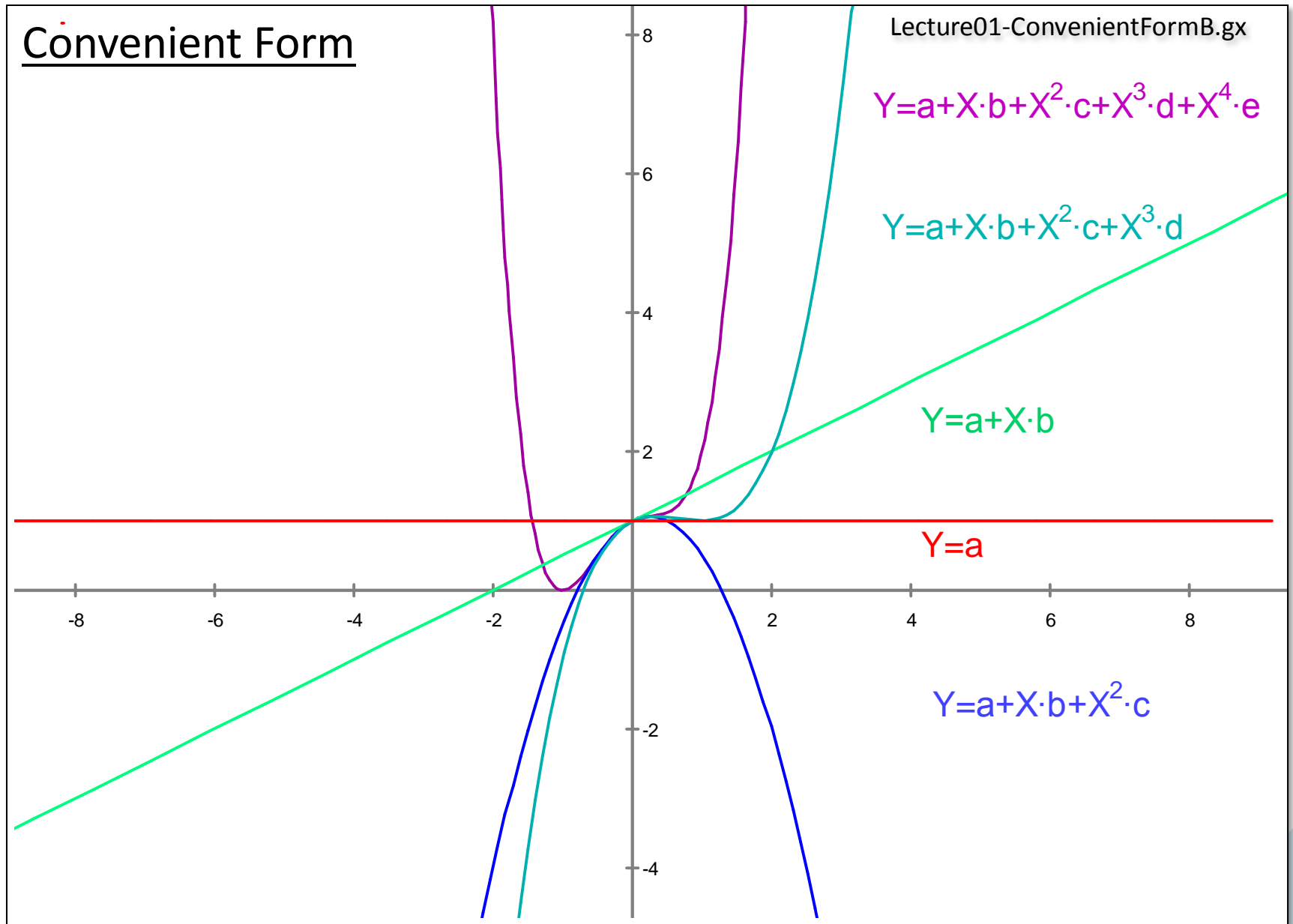


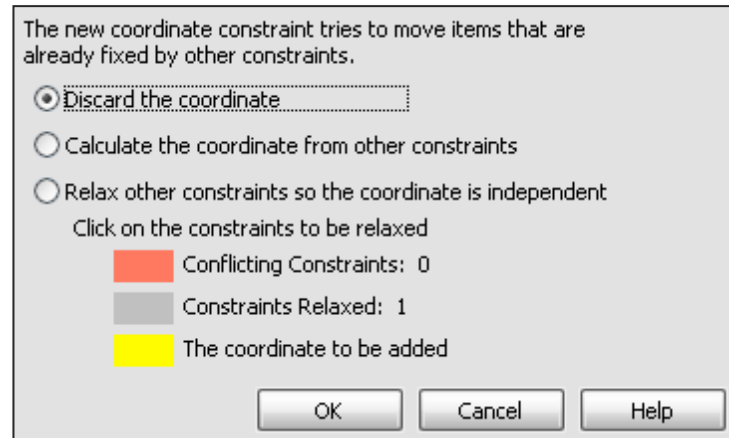
Labeling



- 1) Click Draw→Text.
The cursor will change to a this: +A
- 2) Select a region as shown.
- 3) Enter text in dialog box. →
- 4) Click OK
- 5) Double-Click text to edit it.
- 6) Right-Click text to change its display properties.







This dialog appears when a drawing is “overconstrained”.

“Blank Looks Are Still Free”

A Popular Explicit Function

The explicit function:

$$y = A \cdot \sin(\omega \cdot t + \phi);$$

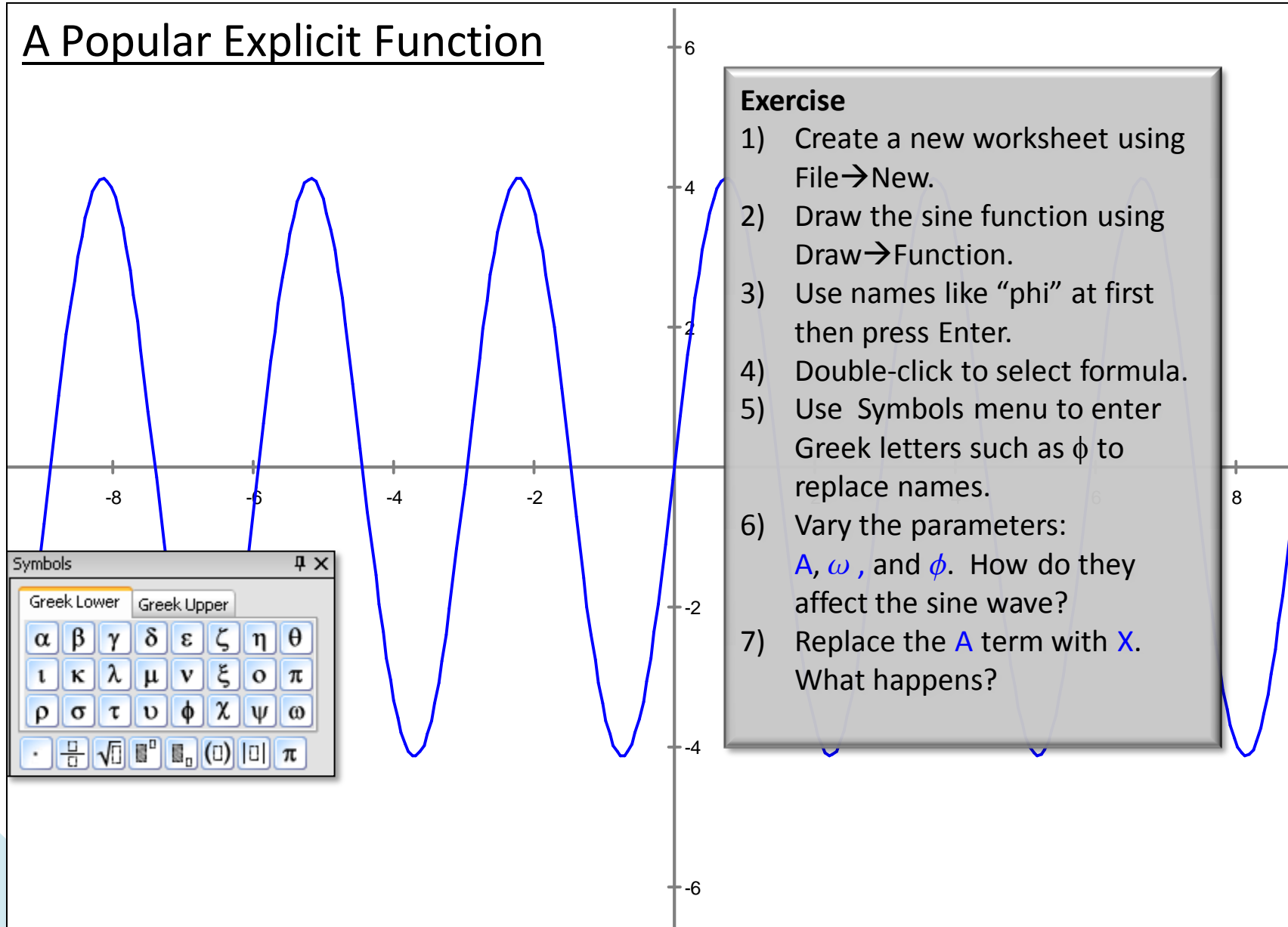
occurs frequently in mathematics, physics, and electronics.

It says that y is a function of A , ω , t , and ϕ . These one-letter symbols originated in the days where equations were written by hand, on chalkboards, and economy of communication was the priority. A stands for Amplitude, ω for frequency, t for time and ϕ for phase. We could just as well have written:

$$y = \text{Amplitude} \cdot \text{sine}(\text{frequency} \cdot \text{time} + \text{phase});$$

Amplitude, *frequency* and *phase* are taken to be constants, but they need not. They can change with time to model many natural phenomena. When using the Draw→Function tool, we will use X to stand for time as the next page shows:

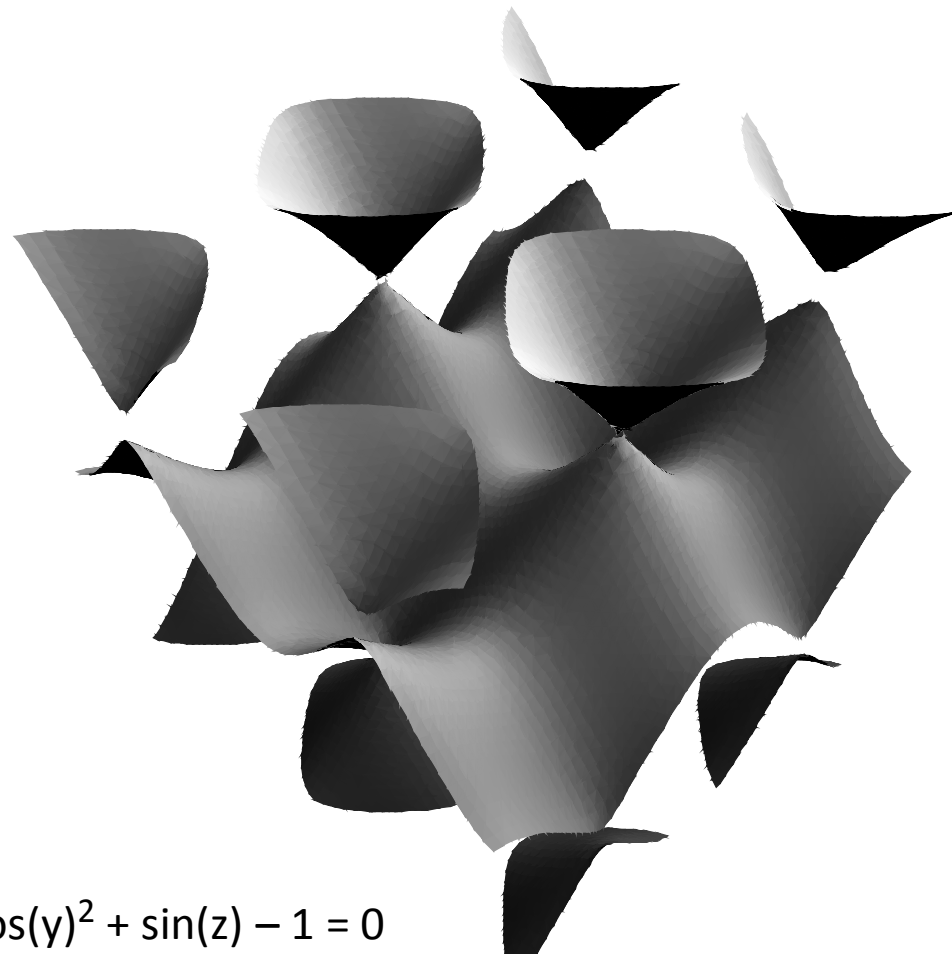
A Popular Explicit Function



Exercise

- 1) Create a new worksheet using File→New.
- 2) Draw the sine function using Draw→Function.
- 3) Use names like “phi” at first then press Enter.
- 4) Double-click to select formula.
- 5) Use Symbols menu to enter Greek letters such as ϕ to replace names.
- 6) Vary the parameters:
 A , ω , and ϕ . How do they affect the sine wave?
- 7) Replace the A term with X . What happens?

Implicit Equations



$$\sin(x)^2 + \cos(y)^2 + \sin(z) - 1 = 0$$

Implicit Equations

Sometimes we encounter a complex equation that can't be explicitly solved for the variable we want. When it is not possible to isolate the variable we want we use the term **equation** rather than **function**. More on that later.

An implicit equation is an algebraic curve formed by points that satisfy an equation. Implicits TEST their input X and Y values and return true if the implicit relationship is satisfied. The general solution of implicit equations requires a search! When we have any equation of the form:

$$f(x, y, z, \dots, a, b, c \dots) = 0$$

We say that the equation is implicit. Setting any expression equal to zero makes it implicit, even if it was explicit before. A good example of an implicit equation is the equation of the circle:

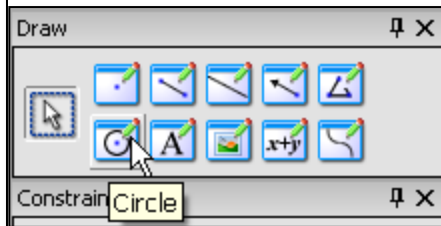
$$x^2 + y^2 - r^2 = 0$$

It is not possible to solve this equation **explicitly** for either x or y and obtain the entire circle.

However, GX™ provides us tools to draw implicit equations.

Circle Drawing

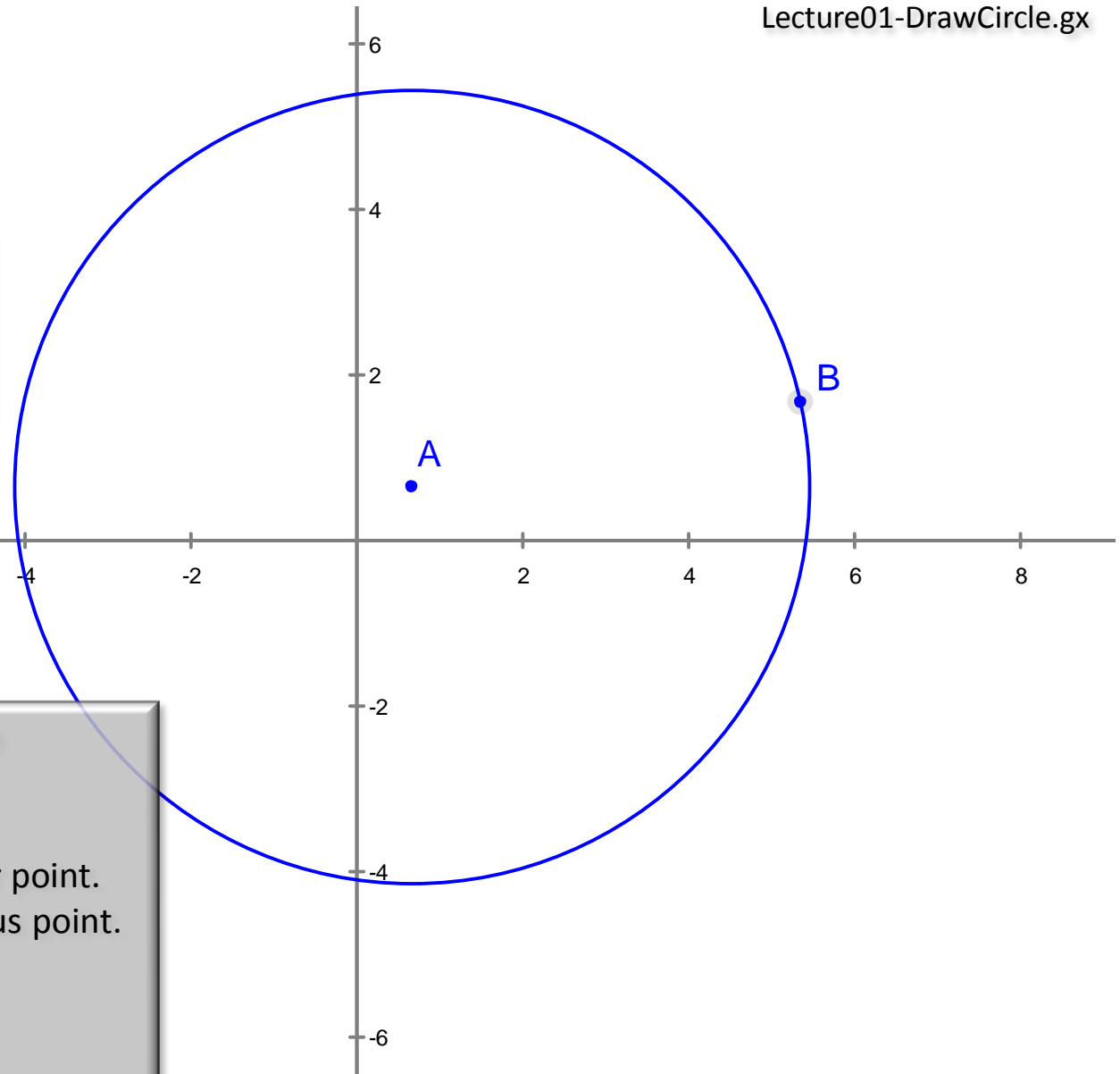
Lecture01-DrawCircle.gx



Steps in Drawing A Circle

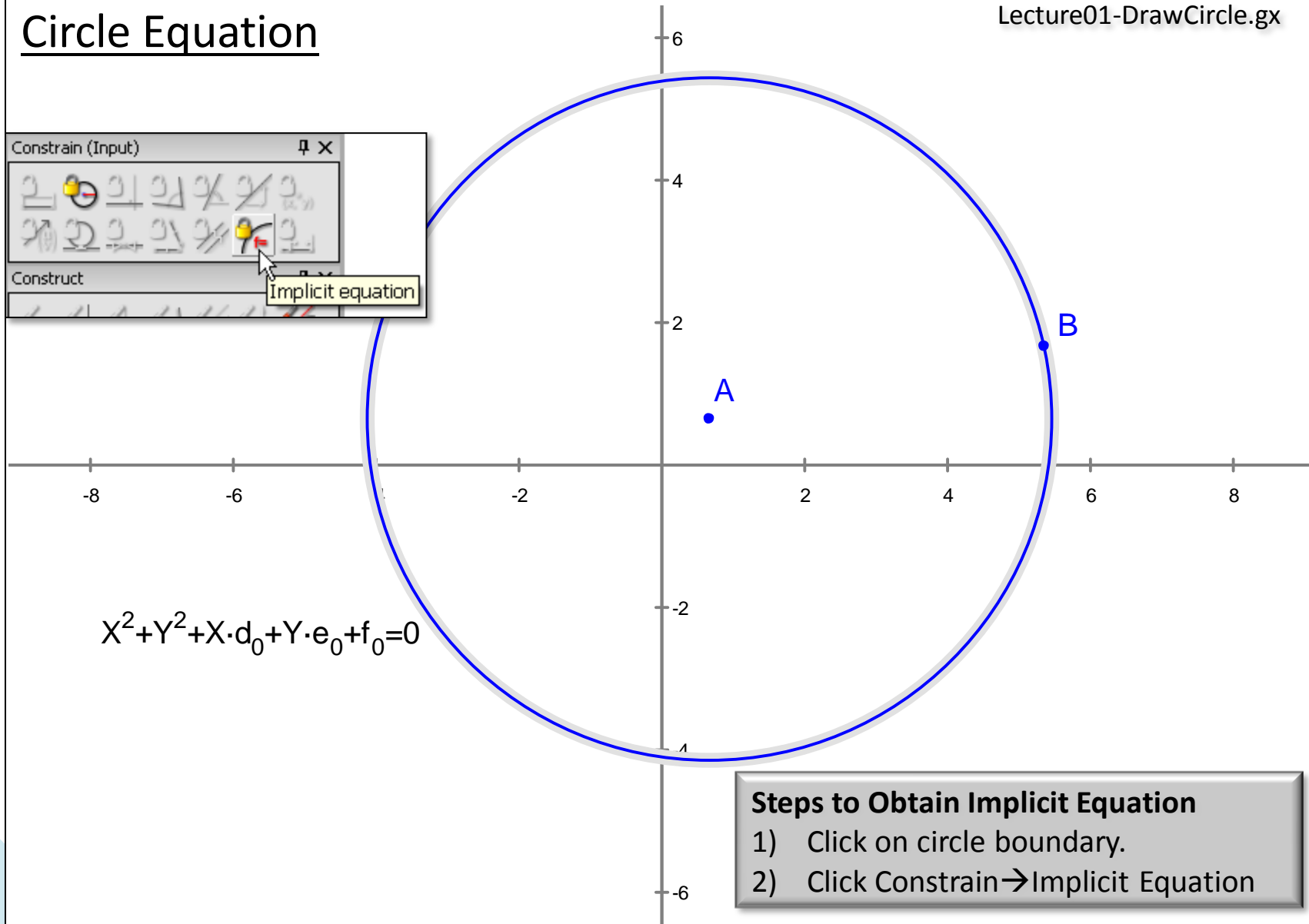
- 1) File → New
- 2) Click Draw → Circle
- 3) Click to create center point.
- 4) Click to create a radius point.
- 5) Press ESC

The circle appears.



Circle Equation

Lecture01-DrawCircle.gx

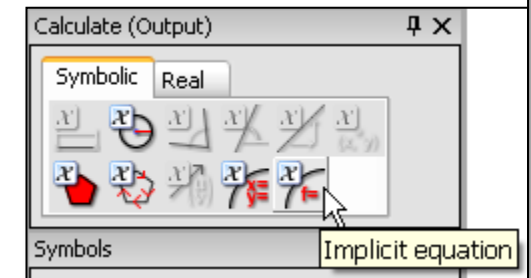
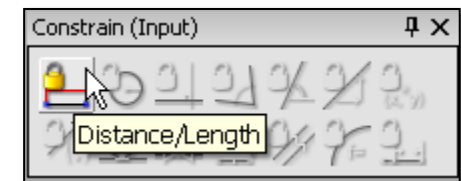
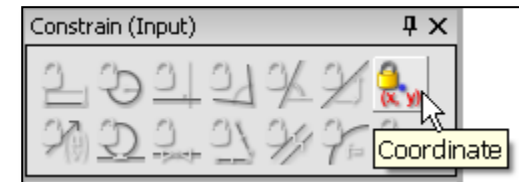
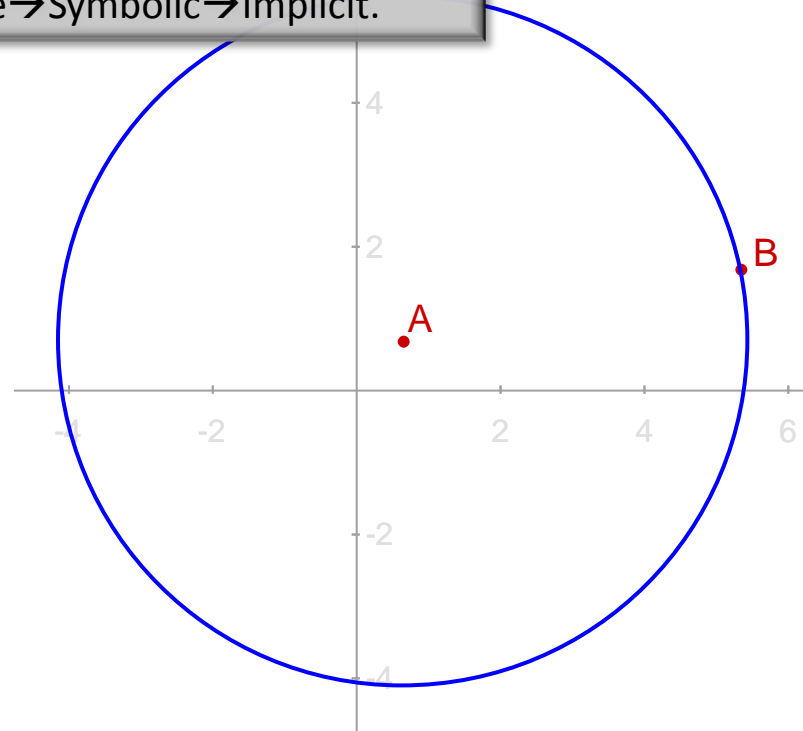


General Circle Equation

Lecture01-CircleDrawing.gx

Steps to Calculate Implicit Equation

- 1) Select Point A.
- 2) Constrain its Coordinate as (h, k).
- 3) Select Point A and Point B.
- 4) Constrain its Distance as r.
- 5) Calculate \rightarrow Symbolic \rightarrow Implicit.



Variables		
Variables		
Name	Value	Locked
h	0.66666667	-
k	0.64583333	-
r	4.7973698	-

The General Form of a Circle

... enables one to specify the radius and center (h, k) . This general form is:

$$(x - h)^2 + (y - k)^2 - r^2 = 0$$

But this doesn't look anything like the form that GX™ gave us...

$$X^2 + Y^2 + X \cdot d_0 + Y \cdot e_0 + f_0 = 0$$

until we expand the first equation like so:

$$x^2 + y^2 + x(-2h) + y(-2k) + (h^2 + k^2) - r^2 = 0$$

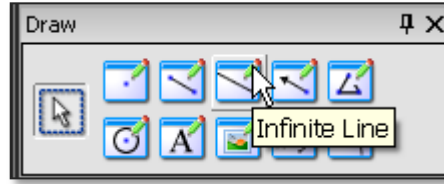
and compare similar terms. Now we can read off coefficients:

$$d_0 = -2h; \quad e_0 = -2k; \quad f_0 = h^2 + k^2 - r^2;$$

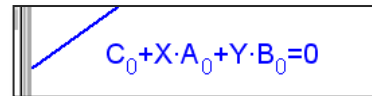
In later chapters we will dive further into the fantastic properties of implicit functions, a more plentiful class than their explicit counterparts.

Exercises

- 1) Draw an infinite line using the tool, Draw→Infinite Line:



- 2) Use the Constrain→Implicit Curve tool to produce the equation of the line:



- 3) The implicit formula for an explicit line translated by a distance (h,k) from the origin is:

$$m(x - h) + b - (y - k) = 0;$$

Using the technique described for the circle, how do the coefficients of the implicit line relate to the GX™ form. That is, what are the values of C_0 , A_0 and B_0 in terms of m , b , h and k ?

- 4) Animate each of the variables in the Variables Dialog and record the effect they have on the implicit version of the infinite line. How do these compare with the rise and run animation done at the beginning of the book?

The Ladder

There is a fascinating progression and yes, its musical. An explicit function can always be converted to an implicit function, but not the other way round about.

The implicit form of $y = a + bx$ is: $a + bx + cy = 0$

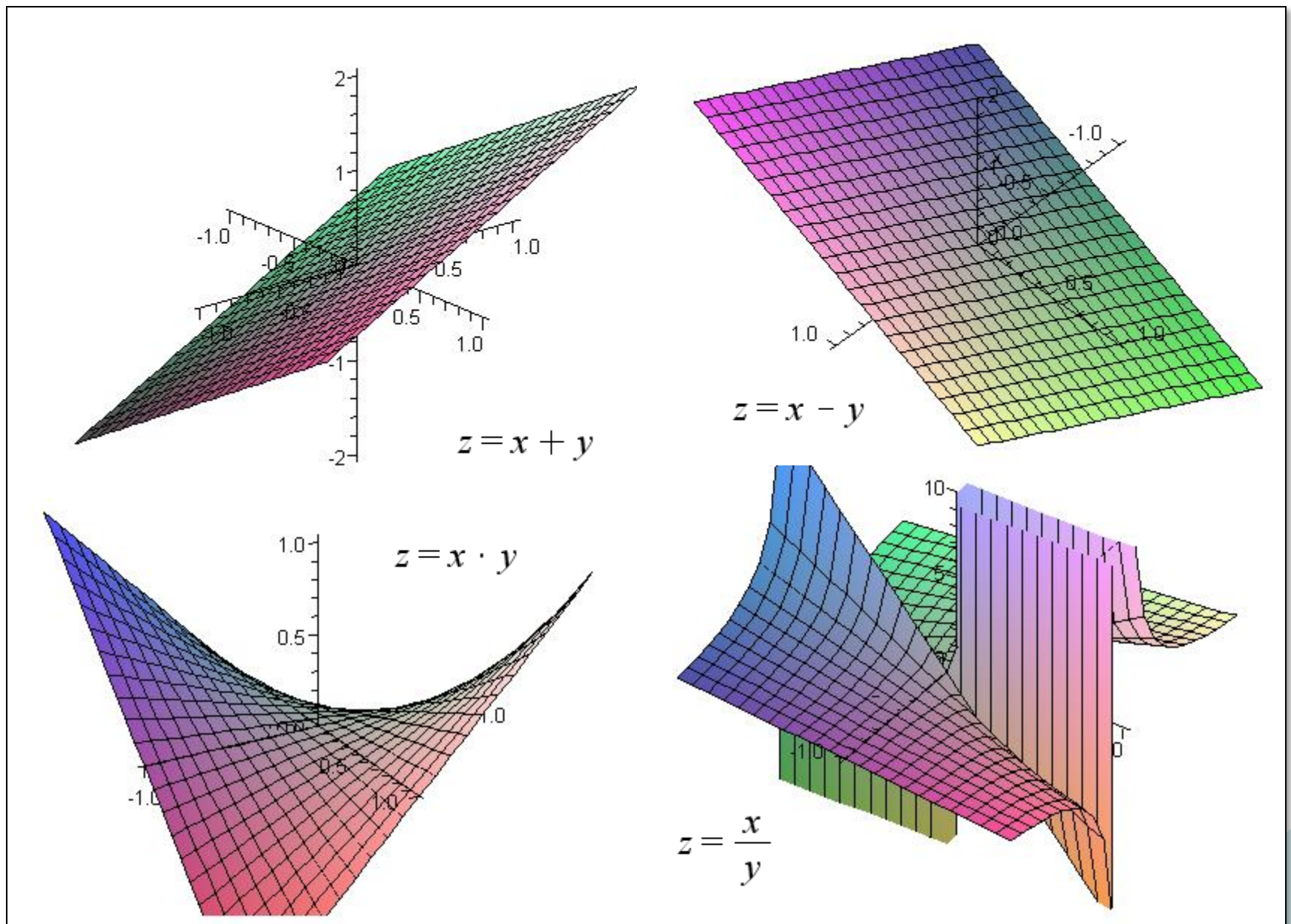
and this can be made a topographic surface:

$$z = a + bx + cy$$

We find the contours of this topographic tale by setting $z = 0$. The “roots”, are “zero-crossings” of the x - y surface at the plane $z=0$. These roots are curves. We can do this for the four basic operations of mathematics:

$$z = x + y \quad z = x - y \quad z = x \cdot y \quad z = \frac{x}{y}$$

The constants b and c in front of x and y scale or distort these graphs. The constant a acts to move them up and down by a fixed amount.



The Rungs

Perhaps you perceive the progression. Here are a few rungs in the ladder:

$$\begin{aligned}ax + by &= 0 \\ax + by &= z \\ax + by + cz &= 0 \\ax + by + cz &= w \\ax + by + cz + dw &= 0\end{aligned}$$

Each rung leads to a higher dimensional space. The implicit equation we are currently using is but a contour plot for an explicit equation in the next higher dimension. It goes on like a ladder forever!

Implicit equations are contour plots, and require a search to discover.

Explicit equations are deterministic and can simply be drawn.

So we connect a search, with a deterministic guarantee of solution.

So now you have seen the ladder, and its pretty cool, don't you think?

Parametric Equations

We have discussed **explicit functions** where a variable on the left is expressed in terms on the right hand side of the equation. We covered **implicit functions**, algebraic curves defined by setting a collection of terms equal to zero. We will now describe **parametric functions**, which could also be called **generating functions** – they generate coordinate pairs as output, given parameter values as input.

In a parametric equation each geometric coordinates, x , y or z is written in terms of a parameter, often named s , t , or θ . A parametric equation is really a set of equations, one for each coordinate we are drawing, x , y , z , etc. Since we are doing two-dimensional geometry in the plane we will just use x and y . We **could** instantly loft our curves into space by specifying one more function for z . The parametric equations for a circle are:

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

In these as we vary the parameter θ from 0 to π , we **GENERATE** x and y coordinates for the top half of a circle. If we run θ from 0 to 2π we get the whole circle. We can stop short or keep retracing the curve. Compare the three representations for a circle in the next slide:

Three Forms of a Circle

Representations of A Circle:

Explicit

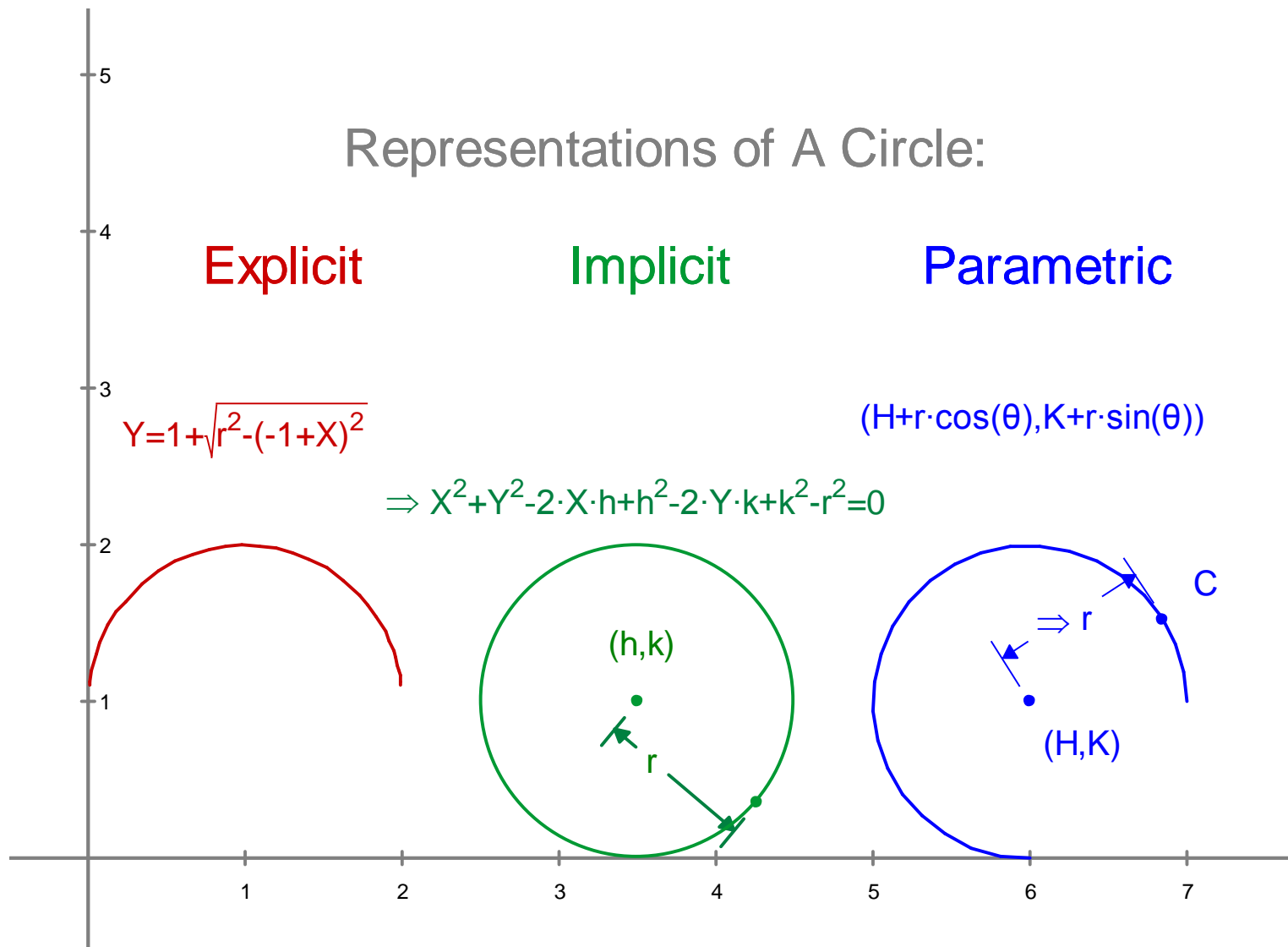
$$Y = 1 + \sqrt{r^2 - (-1 + X)^2}$$

Implicit

$$\Rightarrow X^2 + Y^2 - 2 \cdot X \cdot h + h^2 - 2 \cdot Y \cdot k + k^2 - r^2 = 0$$

Parametric

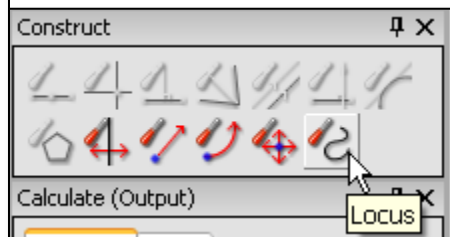
$$(H + r \cdot \cos(\theta), K + r \cdot \sin(\theta))$$



Parametric Curve Drawing

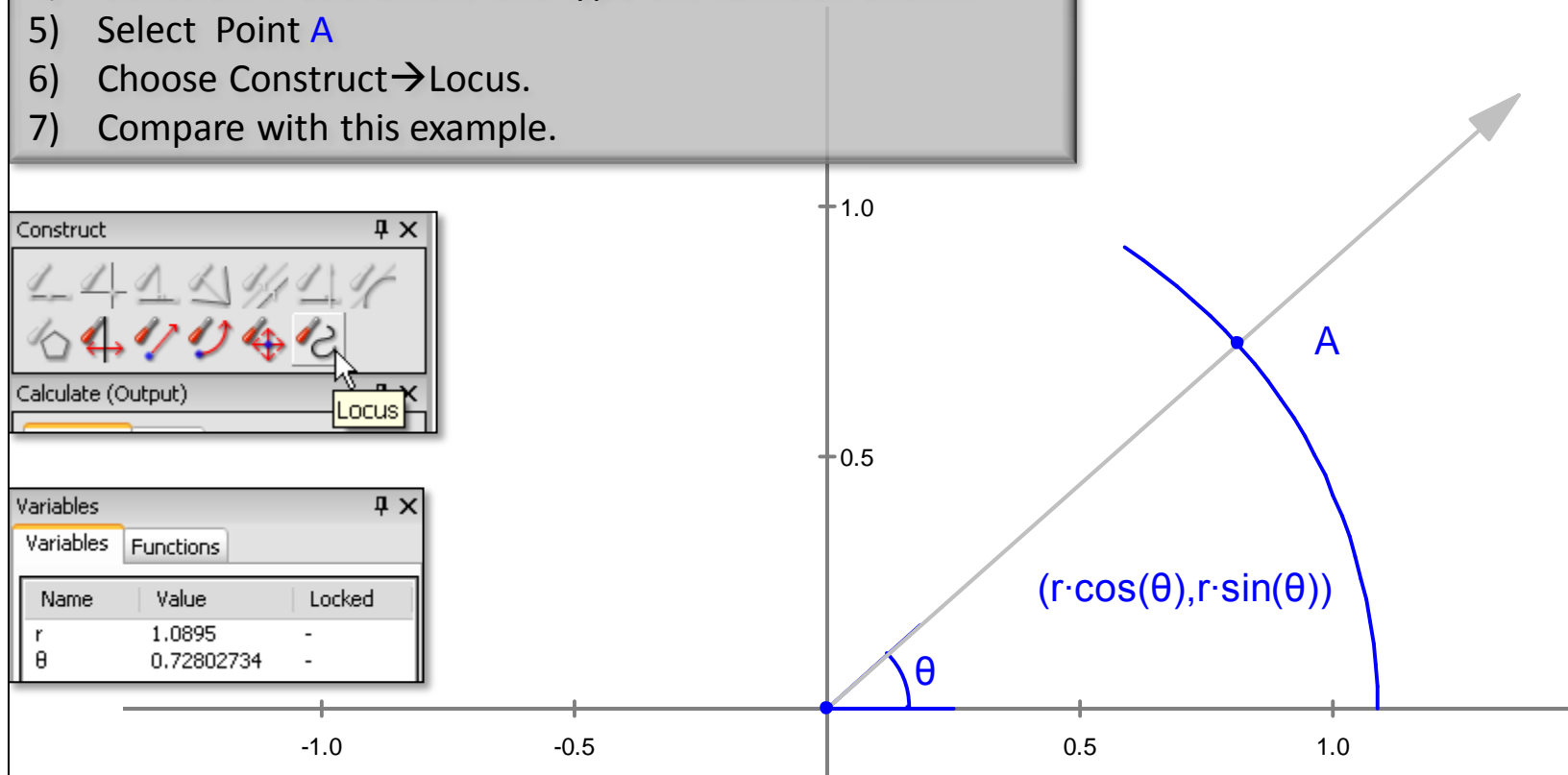
EXERCISE

- 1) Create a new worksheet.
- 2) Create Point A. Press ESC.
- 3) Select Point A
- 4) Constrain \rightarrow Coordinate and type the **function shown**.
- 5) Select Point A
- 6) Choose Construct \rightarrow Locus.
- 7) Compare with this example.



Variables

Name	Value	Locked
r	1.0895	-
θ	0.72802734	-



Parametric Equations

In the case above, the parameter θ had an intuitive geometric meaning – the angle between the x -axis and a ray through a point on the circle. This is not always true. Great effort has been expended to make parameters intuitive, as in “arc-length parameterizations”.

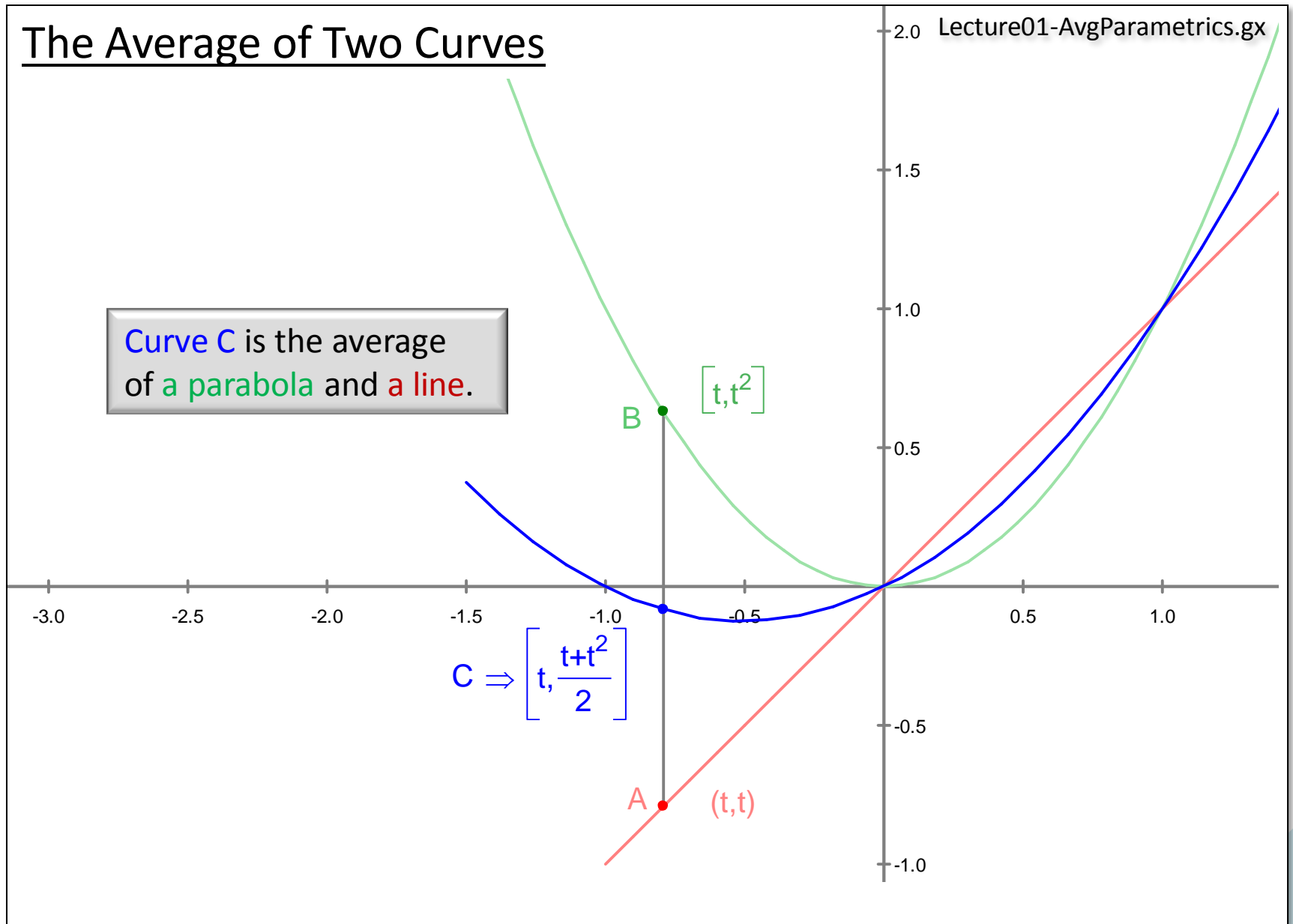
A common parameter name is s , for the arc length of the path being drawn. We might use the parameter t , to represent time. Whatever makes the point clear. Unlike their implicit cousins which provide a test of truth but no method, parametric equations are **generators** of paths through space. Though each equation is a function, the generated curve can loop over itself and thus may not be a function. They enable flexible shape generation with the guarantee of being able to compute an answer more rapidly than an implicit search.

We write the coordinate pair (x, y) as a pair of functions $(r \cos(\theta), r \sin(\theta))$.

Parametric functions are often called **vector-valued functions**, since a function is required for each coordinate we are specifying. All explicit functions can be represented as parametric functions. The variety of parametric functions one can generate with GX™ is truly amazing.

For example. If we draw a line between a point on a parabola and a point on a line, what is the curve traced out by the midpoint? Is it linear? It is parabolic? If we average the two what shape do we obtain?

The Average of Two Curves



Parametric Power Tools

What we just did has a very rich set of possibilities, and we started with a single point! By specifying the behavior of one point, we can, using the locus tool, see the family of points generated by the parametric equations. You could stop now and enjoy an entire career of studying the behavior and utility of these equations. But wait... there's more. What happens if we assign two points A, and B, their own independent functions? A and B don't know about each other, and each has their own personal locus.

But if we draw a line segment to connect A and B something happens. The line segment AB has to serve two masters. Line AB has to satisfy the A functions at the A end, AND it must also obey the B functions at the B end! Line AB is pulled in two directions at once. If you've ever felt jerked around, this is your equation!

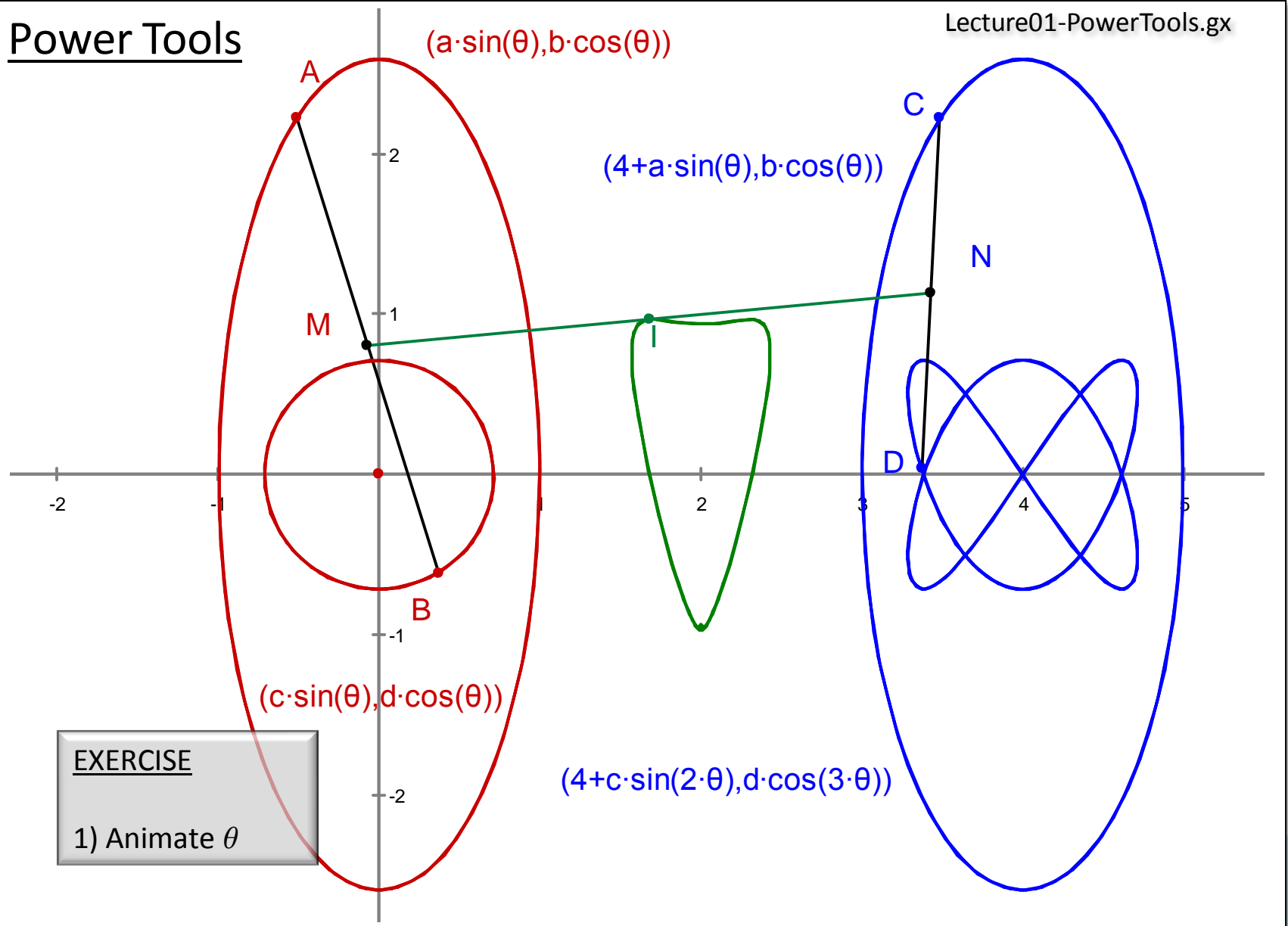
This is a powerful idea in life and in math. There is one point on the line segment that will evenly blend the effects of its two endpoints and that point is the midpoint.

We can compound the expressive richness of this idea.

We could have another line segment driven by its own points C, and D and ask what the relationship is between the midpoint of the line segment connecting the two line segments. The possibilities are endless.

Power Tools

Lecture01-PowerTools.gx



EXERCISE

1) Animate θ

End